DOCUMENT RESUME

ED 069 496 | 24 | SE 015 142

AUTHOR	Lankford, Francis G., Jr.
TITLE	Some Computational Strategies of Seventh Grade Pupils. Final Report.
INSTITUTION	Virginia Univ., Charlottesville. School of Education.
SPONS AGENCY	National Center for Educational Research and Development (DHEW/OE), Washington, D.C.
BUREAU NO	BR-2-C-013
PUB DATE	Oct 72
GRANT	OEG-3-72-0035
NOTE	96p.
EDRS PRICE	MF-$0.65 HC-$3.29
DESCRIPTORS	*Algorithms; Fractions; Grade 7; Learning; *Mathematics Education; *Research; Secondary School Mathematics; Whole Numbers
IDENTIFIERS	*Computation; Diagnostic Interviews

ABSTRACT

One hundred seventy-six seventh grade students underwent a recorded interview where each was given a set of computational exercises and asked to say aloud his thinking as he worked them. The most frequently used strategies in computations with whole numbers and fractions are described in detail, an analysis of the nature of wrong answers is included, and characteristics of good and poor computers are listed and discussed. Thirteen conclusions are given, covering computational strategies, vertical vs. horizontal problem arrangement, mathematical vocabulary of students, estimating answers, and the technique of using recorded interviews in research. The computation problems given to the students are included in the report, and the appendices list all the wrong answers given with the accompanying verbal description by the student. (DT)

Final Report

Project number 2-C-013

Grant number OEG.-3-72-0035

Francis G. Lankford, Jr.
School of Education
University of Virginia
Charlottesville, Virginia, 22903

Some Computational Strategies of Seventh Grade Pupils

October, 1972

U.S. Department of Health, Education, and Welfare
Office of Education
National Center for Educational Research and Development
(Regional Research Program)

and

The Center for Advanced Study
The University of Virginia.

FINAL REPORT

Project Number 2-C-013
Grant Number OEG-3-72-0035

SOME COMPUTATIONAL STRATEGIES OF SEVENTH GRADE PUPILS

Francis G. Lankford, Jr.
University of Virginia
Charlottesville, Virginia 22903

October 1972

U.S. Department of Health, Education, and Welfare
Office of Education
National Center for Educational Research and Development
and
The Center for Advanced Studies, The University of Virginia

ACKNOWLEDGEMENTS

The principal investigator received much help and support in the work on the project reported here. The Center for Advanced Studies at The University of Virginia awarded an associateship which made it possible to devote an entire semester to the project. The U. S. Office of Education granted funds, through its Regional Research program, which extended the time that could be devoted to the project and covered other expenses.

The teachers and principals in the several schools arranged places for the pupil interviews and carefully scheduled the pupils to appear one at a time. Central office officials gave the essential approval to use the schools.

Many colleagues in the School of Education at the University gave both help and encouragement. Special thanks are expressed to Dean Frederick R. Cyphert, and to Professors Richard M. Brandt, Milton D. Jacobson, Donald Medley, William H. Seawell, William C. Lowry, Donald H. Shoemaker, Earl P. Smith, as well as to research assistants William Ware and James Muir. Mr. Rod Collins in the Radio and Television Center at The University of Virginia was especially helpful in preparing taped records of samples of the pupil interviews.

Professor Joseph N. Payne at the University of Michigan gave valuable suggestions on the design and conduct of the study. He also read the first draft of the report and offered helpful reactions.

Grateful acknowledgement of all the help that was so generously given is sincerely expressed.

TABLE OF CONTENTS

SOME COMPUTATIONAL STRATEGIES OF SEVENTH GRADE PUPILS

Introduction

It is encouraging that currently there is widespread interest in improving the computational skills of pupils--especially in the elementary and early secondary years. This interest has found expression in numerous experimental efforts to individualize instruction in computational skills. It may have been stimulated by the calls to "accountability" in the schools and the adoption of "performance contracting." The relatively low scores of many pupils on the computational skills part of standardized achievement test batteries which have been seen in many schools may also have supported this interest.

Many of the experimental programs have started with a detailed analysis of the tasks, or skills, in computation--placing them in many categories related to the content of the operations involved.[1] Often the next step in the design of a pre-test to locate the particular tasks or skills a pupil does incorrectly. Remedial exercises--associated with variously designed programmed instruction--are keyed to the pre-test. Then follows a post-test--similar to the pre-test--intended to show whether or not the pupil can now perform at a satisfactory level.

---

[1]An example may be taken from the work of Patrick Suppes of Stanford University who has done extensive research in computer assisted Instruction. In a Teacher's Handbook for CAI Courses (Technical Report No. 178, Sept. 1, 1971, Institute for Mathematical Studies in the Social Sciences, Stanford University) 14 "strands" in arithmetic are listed. One of these is the "Horizontal Subtraction Strand." The content of this strand is summarized as follows. (P. 13)

Differences in canonical and noncanonical[a] format.
  Maximum minuend is 9
  Maximum minuend is 10
Problems with two-digit minuends in canonical format
Problems with one-digit minuends in noncanonical format
Problems with two-digit minuends equal to or less than 19 in noncanonical format
Problems with minuends greater than 19 in noncanonical format
Problems with a difference on both sides of equal sign

(a) The problem 4 - 0 = _?_ is in canonical format
The problem 7 - _?_ = 7 is in noncanonical format.

An example of an experiment in such an individualized program is the Individualized Mathematics System[1] developed at The Regional Education Laboratory for the Carolinas and Virginia. In this system the elementary school curriculum is divided into 11 topics; each topic is divided into nine levels of difficulty, and for each level of difficulty there are a number of specific skills. "By dividing the work into units of skills, mathematics is presented as a series of small, related tasks. . . ."

A placement test determines the topic on which a pupil begins work, and at what level of difficulty. "Progress is evaluated by short check-up tests following each skill folder, and then by a post-test covering all the skills assigned for the unit."

Commercially available diagnostic tests are also often arranged by type or class of computational exercise. Scores may reveal, for example, that a pupil is weak in adding fractions of any one or more of these classes: (1) denominators the same, (2) one denominator, the common denominator, (3) neither denominator, the common denominator, (4) mixed number plus a fraction, (5) mixed number plus mixed number.

The study reported here was prompted by the belief that pupils vary not only in the types of computational exercises they can do successfully but also in the computational strategies they employ. It was thought that these strategies are highly individualized and that they are often not revealed in the pencil and paper answers to test items or diagnostic exercises. If some insight could be gained into the thinking sequence followed by a pupil as he computes, perhaps some clues might be obtained as to when and how this thinking becomes faulty. It was expected that such "patterns of thinking" might often be quite unorthodox but for a particular pupil a satisfactory substitute for orthodoxy; while for another pupil a favorite strategy could be faulty arithmetically and destined to yield incorrect answers. It was further expected that computational strategies of successful computers differ substantially from those of unsuccessful computers.

Problems with a regroup in canonical format
Problems without a regroup in noncanonical format
Problems with difference on both sides of equal sign.

[1]The Individualized Mathematics System. Regional Education Laboratory for the Carolinas and Virginia. Chapel Hill and Duke Streets, Durham, North Carolina.

In brief this study was designed to examine these basic assumptions.

1. Patterns of thinking--computational strategies--which pupils develop in their study of elementary mathematics are highly individualized and often do not follow the orthodox models of textbook and classroom.

2. There are observable differences in the patterns of thinking--computational strategies--of successful computers and unsuccessful ones.

3. Clues for remedial teaching of computational skills may be derived from an examination of the patterns of thinking--computational strategies--of pupils who are unsuccessful computers.

In the original design of this study it was hoped that this question could also be examined, i.e., what effects on patterns of thinking--computational strategies--are resulting from some of the current efforts to teach low achievers in mathematics? Limitations of time and manpower made it impossible to seek an answer to this question in this study. It is hoped that it can be explored in a subsequent study.

Related Research

Over the fourteen year period in 1957-1970 there appeared annually in The Arithmetic Teacher a list of research, completed the previous year, on elementary mathematics, grades K-8. In 1971 this series was continued in the Journal for Research in Mathematics Education--including items of research at the secondary level.[1] In all, these 11 lists contained 931 items. These lists together with the annotations were examined for the years 1960 through 1970[2] in an effort to locate studies that used individual pupil interviews to analyze computational strategies of pupils.

---

[1] Suydam, Marilyn N. and Weaver, J. Fred. "Research in Mathematics Education (K-12)." Reported in 1970 Journal for Research in Mathematics Education, Vol. 2, No. 4, November, 1972, pp. 257-299.

[2] Weaver, J. Fred. "Research on Elementary School Mathematics," Arithmetic Teacher. For 1960 in May, 1961, pp. 255-260 and October, 1961, pp. 301-307; for 1961 in May, 1962, pp. 287-290; for 1962 in May, 1963, pp. 297-300; for 1963 in April, 1964, pp. 273-275; for 1964 in May, 1965, pp. 382-387 and November, 1965,

in addition all the quarterly issues of The Journal of Research in Mathematics Education since its inception in January, 1970 were reviewed.

The research most closely related to the study reported here was a dissertation completed by Miller[1] at Indiana University in 1960. She recorded interviews with 40 sixth grade pupils and analyzed the responses in four categories. There was no detailed analysis of the computational strategies used. There have been diagnostic studies intended to identify computational strategies but using written responses of pupils instead of both oral and written responses as in the study reported here. An example is the study reported by Roberts.[2] He examined the written work of selected third grade pupils "(1) To determine the computational skills in which Third Grade Pupils were most deficient, and (2) To discover if any generalizations could be derived about pupils' failure strategies."

Admittedly this review of research has been more extensive than intensive. If a more careful review had been possible, doubtless other related studies might have been located. However, it seems safe to assert that the study here reported does not duplicate any research listed or reported in the sources mentioned above for the period 1960 through 1970.

---

pp. 577-578; for 1965 in May, 1966, pp. 414-427; for 1966 in October, 1967, pp. 509-517; Reidesel, C. Allan, Suydam, Marilyn N. and Pikaart, Lin, "Research on Mathematics Education K-8 for 1967," Arithmetic Teacher, October, 1968, pp. 531-544; Reidesel, C. Allan and Suydam, Marilyn N., "Research on Mathematics Education Grades K-8 for 1968," Arithmetic Teacher, October, 1969, pp. 467-478; Suydam, Marilyn N., "Research on Mathematics Education Grades K-8 for 1969," Arithmetic Teacher, October, 1970, pp. 511-527.

[1]Miller, Frances Pauline, "An Analysis of Sixth Grade Pupils' Thinking Regarding Their Solution of Certain Verbal Arithmetic Problems," Dissertation Abstracts, Vol. 21, September, 1960, p. 503.

[2]Roberts, G. H., "The Failure Strategies of Third Grade Arithmetic Pupils," Arithmetic Teacher, Vol. XV, May, 1968, pp. 442-446.

Methods and Procedures

What is being called a "diagnostic interview" was the method employed for obtaining basic data used in this study. In a room where pupil and interviewer may work alone and undisturbed the pupil is asked to do a set of computational exercises as he usually does them but to say aloud his thinking as he computes. The explanation of what is requested of the pupil goes something like this.

> "On these sheets are some exercises in arithmetic. They are just like those you have done many times--in earlier grades--or even in class this year. I want you to do them exactly as you usually do when you do homework or have seatwork in class. If you use scratch paper, there is plenty of space on these sheets for any scratchwork you wish to do. If any of these exercises are written in a way that you are not accustomed to, rewrite them so they are in a form that you like to work with. The only new thing I ask you to do is to say out loud what you are thinking as you do your work. Just 'talk to yourself as you work.' I suspect that you often do this anyway when you are working alone. I do. My reason for wanting 'to hear what you are thinking' is that if you get an incorrect answer I can better decide just where your thinking went wrong and perhaps can suggest some help after we are through. Even if you don't make any mistakes, I still want to know what your thinking is for this may or may not be what you should continue to use. Now let's start with this first exercise. What does it say to do? Read it and do it."

Experience has indicated that better results are obtained from the interviews if these simple guides are followed:

1. A verbatim recording of the interview is made on cassette tape. A recorder with built in microphone is preferred. It should be in place before the pupil comes for the interview. After the preliminary explanations of procedure--described above--the recorder should be turned on without comment. To refer to the recorder in any way proves to be distracting. Otherwise the pupil soon forgets it is present.

2. Let the pupil's first activity be to write his name and date. Scrupulously avoid any use of the pupil's name in the interview. This means that the tape can be used later completely anonymously. Label the tape with the pupil's name, or otherwise, so the oral record can later be matched with the written record and other data such as test results.

3. Give a pupil the preferred seat at a desk or table. The interviewer should sit on the left side of a right-handed pupil and the right side of a left-handed pupil. This provides easier view of his work.

4. As the pupil starts, have him read the first exercise aloud. This helps him to start talking comfortably. Let the first exercise be simple enough that he is almost sure to get the correct answer. Compliment him on his correct answer. It may be necessary to give the pupil further instructions as he proceeds such as "speak a little louder" or "be sure to tell me each step you take."

5. Have the pupil read the caption and each exercise before he undertakes it. This gives a good record of his vocabulary and how his reading of the exercise is related to the computational strategy he uses.

6. Excessive directions can be distracting. It is better not to interrupt the pupil as he computes. When he has finished and written an answer it is better then to ask him questions that require him to clarify or elaborate his thinking.

7. Carefully observe points when the pupil hesitates and does some mental calculation silently before proceeding further. <u>After</u> he has finished the exercise, go back to these points and ask for an oral report of his thinking. For example, in column addition a pupil may say "8 and 8 are 16 plus 1 is 17 and hesitate as he adds 17 + 9 to get 26." When asked "how did you get 17 and 9 to be 26?" he may reply "I thought 17 + 3 is 20 and 6 is 26" or "I thought 17 + 10 is 27 and 1 less is 26," or "I thought 9 + 7 is 16 and 1 and 1 are 2."

8. When a pupil makes a mistake at any stage in an operation he should not be interrupted in any way. It must be remembered at all times that a diagnostic interview is not a teaching exercise in which pupils are led to correct their errors through a series of Socratic type questions. He must be left free to proceed in his own way with never a hint that it may be wrong. Some pupils will ask the interviewer "Is that right?" He must evade an answer and encourage the pupil to proceed in his own way. After an exercise is finished, a pupil may be asked to repeat a step in which he made an error. If he repeats the error, it is not likely to have been a careless one. He is not asked to repeat because he made an error, but so the interviewer may understand more clearly what he did.

9. When a pupil discovers an error either on his own as he proceeds or after he finishes and is asked to repeat a step, he usually wants to erase and correct. Instead he should be directed to mark through the error and write the correction to the side. Later when the tape is replayed, the oral record then matches the written record.

10. Avoid giving clues or leading the pupil through a series of questions--program style.

11. When a pupil's words have been indistinct, the interviewer should, after an exercise is completed, repeat the child's words as faithfully as possible to get a usable record.

12. Do not hurry a pupil. Allow him as much time as he wishes on a single exercise. If time must be controlled, do so by reducing the number of exercises the pupil is asked to do.

In addition to the recordings of the oral comments during the interviews the pupils' work papers--including all scratch work--were retained and used in the analysis of responses.

The Pupils Interviewed

The pupils interviewed numbered 176 and were enrolled in six schools located in Richmond, Virginia; Northumberland County, Virginia (a rural county in Eastern Virginia); Atlanta, Georgia; Washington, D. C.; Detroit, Michigan, and Denver, Colorado. All pupils were enrolled in the seventh grade. The interviews were conducted during the months of February, March and April of 1972. In each school a single class, or section, was chosen representative of the mid-range of achievement and ability. Most of these pupils were interviewed. A few were absent during the time of the interviews and could not be included. In one school the class contained a few more pupils than could be interviewed in the week that could be devoted to this school. Whenever all of a class could not be interviewed those who were came in alphabetical or random order. In some of the schools these basic groups were supplemented by a few pupils chosen from other sections to be sure to include a few high achievement pupils and to assure that about the same number of girls as boys were interviewed. The 176 pupils included 83 girls and 93 boys. Racially it included 90 blacks, 85 whites, and 1 Oriental.

Three sets of data were taken (where available) from the pupil cumulative records maintained by the schools. These included the year-end grade in arithmetic assigned by the school for the sixth grade (the year preceding the interviews). Intelligence quotients based on group intelligence tests were available and were tabulated in all but one school. The grade equivalent scores on an arithmetic achievement test were available in all the schools. These grade equivalents were tabulated for the computation part of these tests for all except one school where only concepts and problem solving scores were on record. In this school the problem solving score was tabulated.

No claim is made that these 176 pupils are representative of seventh graders enrolled in all public schools or even of seventh graders enrolled in the six school systems where interviews were held. This study was in no way a survey of the computational skills of typical seventh graders. Instead it was the purpose to obtain some examples of the thinking used in computing, by pupils with a fairly wide range of backgrounds--including boys and girls, blacks and whites, mathematical achievement, ability, type of school (rural and urban) and geographic location. That this was accomplished is indicated in the tables that follow.

The seventh grade was chosen because by the end of the sixth grade all pupils will have been taught to compute with whole numbers and fractions at least once. Many of these seventh graders will likely be taught operations with whole numbers and fractions again, but it was thought that interviews at the seventh grade level would reveal patterns of thinking--computational strategies--as they have been formed at the end of the elementary school and before the reteaching expected in the remaining years of secondary school.

School Number One - 25 Pupils

Arithmetic Grades
In Grade 6

Grades	Pupils
A	0
B	2
C	2
D	0
F	1
N	7
S	9
O	2
N.A.[1]	2
Total	25

(1) Grades were not available.

Total I. Q.[2]

I.Q.	Pupils
100-109	2
90- 99	3
80- 89	9
70- 79	6
N.A.	5
Total	25

(2) Based on California Test of Mental Maturity given in Grade 7 (Fall, 1971).

Arithmetic Achievement
Computation-Grade Equivalent[3]

G.E.	Pupils
6.0-6.9	1
5.0-5.9	10
4.0-4.9	3
3.0-3.9	4
N.A.	7
Total	25

(3) Based on S.R.A. Achievement Battery given in grade 5.

School Number Two - 30 Pupils

Arithmetic Grades
In Grade 6

Grades	Pupils
A	5
B	8
C	13
D	2
F	2
Total	30

Total I.Q.(1)

I.Q.	Pupils
120-129	2
110-119	6
100-109	8
90- 99	6
80- 89	4
70- 79	3
60- 69	1
Total	30

(1) Based on <u>California Test of Mental Maturity</u> given in grade 7 (Fall, 1971)

Arithmetic Achievement
Computation-Grade Equivalent(2)

G.E.	Pupils
9.0 and above	1
8.0-8.9	1
7.0-7.9	1
6.0-6.9	3
5.0-5.9	13
4.0-4.9	5
3.0-3.9	4
N.A.	2
Total	30

(2) Based on S.R.A. Achievement Battery given in grade 6.

School Number Three - 30 Pupils

Arithmetic Grade In Grade 6

Grades	Pupils
A	4
B	9
C	10
D	1
F	2
N.A.	4
Total	30

Total I. Q.[1]

I.Q.	Pupils
120-129	1
110-119	3
100-109	5
90- 99	3
80- 89	7
70- 79	3
N.A.	8
Total	30

(1) Based on California Test of Mental Maturity given in grade 5.

Arithmetic Achievement Computation-Grade Equivalent[2]

G.E.	Pupils
10.0 and above	1
9.0-9.9	1
8.0-8.9	0
7.0-7.9	3
6.0-6.9	9
5.0-5.9	8
4.0-4.9	4
N.A.	4
Total	30

(2) Based on Metropolitan Achievement Test given in grade 6.

School Number Four[1] - 30 Pupils

Arithmetic Grade in Grade 6

Grades	Pupils
A	1
B	6
C	17
D	6
Total	30

Arithmetic Achievement Computation Grade Equivalent[2]

G.E.	Pupils
10.0 and above	1
9.0-9.9	0
8.0-8.9	0
7.0-7.9	3
6.0-6.9	5
5.0-5.9	8
4.0-4.9	10
3.0-3.9	1
N.A.	2
Total	30

(1) No intelligence tests are administered in this school.

(2) Based on Comprehensive Test of Basic Skills given at beginning of Grade 7.

School Number Five - 33 Pupils

Arithmetic Grade in Grade 6

Grades	Pupils
A	10
B	13
C	5
D	1
E	1
N.A.	3
Total	33

Total Intelligence Test Score-Percentile Rank(1)

Percentile Rank	Pupils
90-99	5
80-89	6
70-79	4
60-69	5
50-59	3
40-49	3
30-39	3
20-29	0
10-19	1
0- 9	1
N.A.	2
Total	33

(1) Based on California Test of Mental Maturity given in Grade 5.

Arithmetic Achievement Problem Solving-Grade Equivalent(2)

G.E.	Pupils
6.0-6.9	2
5.0-5.9	12
4.0-4.9	12
3.0-3.9	5
N.A.	2
Total	33

(2) Based on Iowa Test of Basic Skills given in grade 6. There is no separate score on computation from this test.

School Number Six - 28 Pupils

Arithmetic Grade in Grade 6

Grades	Pupils
A	3
B	7
C	11
D	5
F	0
N.A.	2
Total	28

Total I. Q.[1]

I.Q.	Pupils
120-129	4
110-119	3
100-109	8
90- 99	5
80- 89	5
N.A.	3
Total	28

(1) Taken from the cumulative records of pupils at the school on which I.Q. derived from most recent test taken was recorded.

Arithmetic Achievement Computation-Grade Equivalent[2]

G.E.	Pupils
9.0-9.9	1
8.0-8.9	1
7.0-7.9	5
6.0-6.9	6
5.0-5.9	8
4.0-4.9	3
3.0-3.9	1
N.A.	3
Total	28

(2) Based on _Metropolitan Achievement Test_ given in Grade 6.

Exercises Used in Interviews

There were 37 exercises used in the interviews distributed, by type, as follows.

Addition of whole numbers	- 3 exercises
Subtraction of whole numbers	- 3 exercises
Multiplication of whole numbers	- 3 exercises
Division of whole numbers	- 4 exercises
Addition of fractions	- 4 exercises
Subtraction of fractions	- 5 exercises
Multiplication of fractions	- 3 exercises
Division of fractions	- 4 exercises
Fraction comparisons	- 8 exercises

One of the exercises on division of whole numbers was added only for the last two schools where pupils were interviewed. This means that in four schools only 3 exercises in division of whole numbers were used. Also after interviews were completed in four schools, one exercise in subtraction of fractions was deleted and another substituted. This means that no more than 36 exercises were included for any one pupil. These exercises were chosen to require a variety of computational skills and they were arranged in different ways--e.g. both vertically and horizontally. Again the intent was to stimulate pupil thinking over a range of computational exercises and arrangements and to obtain records of examples of this thinking.

The 37 exercises and the way they were arranged for the pupil may be seen on pages 16-18 following.

Length of Interviews

In general the interview with a single pupil did not extend beyond one class period of about 45 to 55 minutes. During the interview the pupil was allowed, for the most part, to take as much time as he wished on an exercise. Occasionally when it was apparent that a pupil was hopelessly confused with an exercise, he would be moved to another to get examples of his thinking on as wide a variety of exercises as time permitted. Many pupils ran out of time before getting to the last 8 exercises with fractions that called for comparisons based on concepts rather than detailed computations. So not every exercise was tried by every pupil. The number of pupils who did not try each exercise and the number of right and wrong answers are all indicated on pages 16-18.

The principal investigator conducted all the interviews in the last four schools and most of those in the first two schools. In these first two schools, he was helped with a few of the interviews by a research assistant thoroughly familiar with the study.

Results

Frequency of Right and Wrong Answers and Omissions

The exercises are listed below and arranged just as they were on the sheets used by the pupils. The number of right answers, the number of wrong answers, the number of omissions, and the percent of attempted exercises with right answers are also included.

The "omit" category must be understood to include those exercises which were not attempted because of limitation of time as well as those which pupils chose not to try because of not knowing how to do them. Something of a combination of these reasons accounted for other omissions. For example, when a pupil used an especially long and tedious strategy for a particular exercise, the interviewer directed him to skip some of the remaining exercises in order to have him try some in each group representing the several operations. Perhaps the most helpful statistic in the table that follows is the one which shows the percent of attempted answers that were correct ones.

	Right	Wrong	Omit	Percent Attempted Exercises With Right Answers
Add				
73 + 24 = ___	168	8	0	96
64 78	169	7	0	96
709 538 291 478	135	41	0	77
Subtract				
93 - 32 = ___	167	9	0	95
86 49	144	32	0	82
703 329	132	43	1	75

	Right	Wrong	Omit	Percent Attempted Exercises With Right Answers
Multiply				
19 x 20 = ___	128	47	1	73
58 75	121	54	1	69
304 506	106	62	8	63
Divide				
27/81	142	25	9	85
48/93	117	48	11	71
15/7590 (two schools)	38	21	2	64
74/6484	91	52	33	64
Add				
3/4 + 5/2 = ___	75	84	17	47
3/8 + 7/8 = ___	97	48	31	67
5 7/8 + 2 1/2 = ___	71	56	49	56
2/3 + 1/2 = ___	68	49	59	58
Subtract				
3/4 - 1/2 = ___	88	65	23	58
8 2/5 4 3/5	90	40	46	69
7 1/2 - 4 1/4 = ___ (4 sch.)	22	17	76	56
5/8 - 1/3 = ___	58	41	77	59
9 2/3 5 7/8 (two schools)	20	24	17	45
Multiply				
2/3 x 3/5 = ___	96	57	23	63
2 1/2 x 6 = ___	48	62	66	44
5 1/2 x 3/4 = ___	38	56	82	40

	Right	Wrong	Omit	Percent Attempted Exercises With Right Answers
Divide				
9/10 ÷ 3/10 = ___	59	85	32	41
15 3/4 ÷ 3/4 = ___	31	73	72	30
6 9/10 ÷ 3 = ___	32	49	95	40
7/8 ÷ 2/3 = ___	31	59	86	34
Which is larger?				
2/3 x 5 or 1 x 5	60	38	78	61
3/2 x 6 or 1 x 6	72	19	85	79
17 ÷ 5/8 or 17 ÷ 1	46	34	96	58
17 ÷ 5/2 or 17 ÷ 1	48	29	99	62
3 9/10 + 7/8 or 3 9/10 + 1	84	22	70	79
8 5/6 + 7/4 or 8 5/6 + 1	63	37	76	63
10 1/9 - 7/8 or 10 1/9 - 1	69	25	82	73
12 3/8 - 5/4 or 12 3/8 - 1	56	35	85	62

There were no omissions for the three addition exercises with whole numbers and very few wrong answers for the first two. There were, however, 41 wrong answers for the column addition of 4 three-digit addends.

There was only one omission for the three subtraction exercises with whole numbers. There were few (9) wrong answers to the first exercise but considerably more (32 and 43) for the other two subtraction exercises.

There were 10 omissions for the three multiplication exercises with whole numbers, and all three had a substantial number (47, 54, 62) of wrong answers.

Considering only the 3 division exercises with whole numbers that were used in all six schools, there were 52 omissions. All three exercises had a considerable number (25, 48, 52) of wrong answers.

It is significant that the number of wrong answers was greatest in division, the next in multiplication, the next in subtraction, and the least in addition.

The omissions increased (17-59) with each exercise among the four addition exercises with fractions. A noticeable fact is the 84 wrong answers for the first of these addition exercises 3/4 + 1/2 = ___.

Three of the subtraction exercises with fractions were used in all six schools. Another was used in the first four schools and was replaced by 9 2/3 - 5 7/8 in the last two schools. Again it is noticeable that 65 pupils got a wrong answer for the exercise 3/4 - 1/2 = ___. The errors were also quite high (24) compared with the correct answers (20) in the two schools where the exercise 9 2/3 - 5 7/8 was used.

The substantially highest number of right answers among the multiplication exercises with fractions was for 2/3 x 3/5; yet there were 57 wrong answers for this exercise. The first of the division exercises with fractions had the largest number (85) of wrong answers and the smallest number (32) of omissions. These were the responses to 9/10 ÷ 3/10 = ___. There were only 31, 32, and 31, respectively, right answers to the remaining three division exercises.

It is to be noticed that the first exercise in each of the four operational groups with fractions had the fewest omissions, and that the number of omissions tended to increase with the remaining exercises in a group. This is explained in large part by the practice of the interviewer to encourage pupils to try the first exercise in each group, at least, if it appeared that time would not permit trying all the exercises. This meant, of course, that the faster and usually better computers tended to be the ones who tried the latter exercises in each group.

There were eight exercises in the final group. Pupils were asked to respond by simply placing a check mark in one of the two blank spaces. Pencil and paper computation was discouraged although many pupils wanted to perform the operations in order to choose an answer thought to be correct. After an answer was checked for each exercise, the pupil was asked to say why he made the choice he did. This explanation was, of course, recorded and used in the analysis presented later in this report. The large numbers of omissions for these exercises resulted from the fact that they came at the end and many pupils did not get to them in the time available for the interview. This again means that this group of exercises were attempted by the faster and, usually, better computers in higher ratio than was the case in the earliest exercises. The highest number (84) of right answers in this group was in response to the question, which is larger 3 9/10 + 7/8 or 3 9/10 + 1? Also this exercise had the smallest number of omissions among those in this group. This probably reflects the practice of the interviewer, followed in the later interviews of starting with this exercise, since pupils seemed to be able to interpret the comparisons based on addition better than those based on the other operations. This exercise, then, tended to give the pupil a better start with this group of exercises than did those based on multiplication and followed by division.

Strategies Frequently Used in Four Operations with Whole Numbers

It is one of the assumptions of this study that computational strategies vary greatly among pupils in the seventh grade. This assumption is abundantly supported by the interviews, as will become clear in the remaining pages of this report. For this reason it is extremely difficult, or almost impossible, to place these strategies in categories without obscuring the individual variations. Despite this fact an attempt was made in the analysis of the interviews to single out a few strategies that were fairly uniform in nature and high in frequency. A presentation of these frequent strategies follows.

Counting was the most frequently used strategy in operations with whole numbers. There were 93 pupils who used counting in the addition of whole numbers; 63 in subtraction of whole numbers; 63 in multiplication of whole numbers and 4 in division of whole numbers. Some pupils (51) used counting to add any digit; others (37) used counting only in "bridging", as in 18 + 6. A few (5) used counting to add large digits only, such as 7, 8, or 9, but did not need to count to add such digits as 1, 2, or 3.

When pupils use counting in an operation, many of them have some means of "keeping count" of the counting. When a pupil thinks, for example, 9 + 8 = ? and says "10, 11, 12, 13, 14, 15, 16, 17," he is likely to have some means to tell him when to stop counting. Most pupils (45) use their fingers for this purpose. Others (14) make motions in the air with a pencil point. The motions are often made in pairs or in triples to help in keeping count. Sometimes (14) dots or marks are made on scratch paper. For example, in 8 + 7 a pupil may make marks in pairs like this as he counts the 7 on to the 8 :::·. Less frequently the marks assume no pattern. For 8 + 7 the pupil merely makes marks like this /////// as he counts 9, 10, 11, 12, 13, 14, 15. Other pupils do not reveal their scheme for "keeping count" of their counting. "I count in my mind" is an explanation given. It is quite likely that some of these pupils do in fact use their fingers but are reluctant to reveal the practice. Many times in the interviews this reluctance to reveal "counting on my fingers" was apparent. Somehow pupils had developed the attitude that if you "count on your fingers" you should keep it to yourself.

Of the 63 pupils who used counting in subtraction of whole numbers, 26 used it with any digit, and 37 used it in "bridging only," as in 16 - 9. It is interesting that 46 pupils counted from the subtrahend to the minuend to get a difference. For 9 - 3, for example, a pupil would say "4, 5, 6, 7, 8, 9, the answer is 6"--probably accumulating the 6 on his fingers as he counted. There were 17 pupils who counted backwards from the minuend to the subtrahend. For example, for 13 - 8 a pupil would say 13, 12, 11, 10, 9, the answer is 5"--again having very likely accumulated the 5 on his fingers as he counted.

Surprisingly there were 63 pupils who used counting in the multiplication of whole numbers. Often (31) counting was used in deriving an unknown combination from a known one. For example, a pupil who did not know the combination 7 x 8 = ? in some instances said "7 x 7 = 49, 50, 51, 52, 53, 54, 55, 56"--stopping with 56 when he had counted 7 fingers on to 49. In other cases the pupil started with 7 x 5 = 35 and counted over 7 fingers 3 times like this "36, 37, 38, 39, 40, 41, 42 : 43, 44, 45, 46, 47, 48, 49 : 50, 51, 52, 53, 54, 55, 56." There were 31 pupils who continued to count when addition was necessary in multiplication. For example, in the product 58 (75) a pupil might say "5 times 8 is 40, put down your 0 and carry the 4 (writing it above the 5 of 58), then 5 x 5 = 25 : 26, 27, 28, 29." Counting was also used in adding partial products by 29 pupils.

A small number (4) of the pupils interviewed used counting to obtain a quotient figure. For example, in 27/81 one pupil wrote 81 marks on scratch paper, counted off three groups of 27 marks each and decided that "27 goes into 81 three times." The next exercise was 48/93. This same pupil extended the 8 marks to 93, counted off 48, then the 45 left, and decided that "48 goes into 93 one time with 45 left over." Pupils continued to use counting as in addition and subtraction when these operations were needed in a division exercise.

Some strategies, other than counting, which were frequently used are next described. One of the three addition exercises with whole numbers was arranged like this: 73 + 24 = ___. Sixty-five pupils first rewrote this exercise vertically. Thirteen of those who found the sum without rewriting added the tens first. That is, 73 + 24 became "7 + 2 is 9 and 3 + 4 is 7."

With high frequency (74) pupils chose, or made, doubles to add. In 64 + 78, for example, pupils would say "8 and 4 are 12 ; 6 and 6 are 12, plus 1 is 13, plus 1 (carried) is 14. In column addition, where the digits in the one's column were 9 + 8 + 1 + 8, pupils would say "8 and 8 are 16, + 1 is 17, + 1 is 18, + 8 is 26" or "8 + 8 is 16, + 1 is 17, + 9 is 26." Another strategy in addition of whole numbers was to work for combinations of ten, or multiples of ten. This was done by skipping about in a column to find two or more addends with a sum of 10 (50), or by subdividing addends to make sums of 10, such as 7 + 5 becoming 5 + 5 + 2. Sometimes two strategies were used as in 9 + 8 + 1 + 8 which would become "9 + 1 is 10, 8 + 8 is 16; and 10 + 16 is 26." Other variations in column addition meant starting with the larger digits in the column (25 pupils), or in some other way following a sequence of addends that was not straight up or down the column (102). Another interesting practice appeared in adding the column 9 + 8 + 1 + 8 which became 17 + 1 = 18 and 18 + 8 = ? In 73 cases this was written on scratch paper aside as $\begin{array}{r}18\\ \underline{8}\\ 26\end{array}$ and the pupil said "8 and 8 is 16, 1 and 1 is 2, the answer is 26." In 33 other cases this was done mentally, without writing the addends separately. In 55 cases pupils added the carried digit in some order other than first or last. That is, this carried digit was worked into the sum whenever the pupil found it most convenient in his thinking.

In the subtraction of whole numbers there was one exercise arranged like this, 93 - 32 = ___. There were 82 pupils who first rewrote this exercise vertically before subtracting. In recent years most pupils have been taught that subtraction is the inverse of addition. That is, 13 - 8 = ? really asks the question 8 + ? = 13. So when a pupil must subtract $\begin{array}{r}86\\ \underline{49}\end{array}$ he should think "9 and 7 is 16; 4 and 3 is 7." Only 27 pupils interviewed showed this pattern of thinking. A much larger number (123) of pupils regrouped quite mechanically. For example, a pupil in the exercise: Subtract $\begin{array}{r}86\\ \underline{49}\end{array}$ would say "mark out the 8, make it a 7; make the 6 a 16" or "take one from 8, add it to the 6." Usually the rewritten exercise would look like this. $\begin{array}{rr}{}^{7}\not{8} & {}^{1}6\\ \underline{4} & \underline{9}\end{array}$ There were 89 pupils who thought of regrouping for subtraction as borrowing. Seventy of these showed the borrowing by rewriting the exercise, while 19 kept the borrowing "in mind." Only 4 pupils thought of the regrouping as changing a hundred to tens and a ten to ones as in subtract $\begin{array}{r}708\\ \underline{329}\end{array}$. These pupils would say something like this "7 hundred become 6 hundred, 0 tens becomes 10 tens: then 10 tens become 9 tens and 8 ones become 18 ones."

A surprising use of quite puzzling vocabulary appeared in the subtraction exercises. Here are some examples in the exercise: subtract $\begin{array}{r}86\\ \underline{49}\end{array}$ "6 can't go into 9; 6 subtract from 9 will be 3." "You can't take 6 from 9 so you borrow 1."

Some examples in the exercise: subtract 708 / 329 are:

> "9 subtract 8 leaves 1";
> "9 won't go into 8; 2 won't go into 0; 3 goes into 7, four times, answer 400."
> "9 from 8 are 1; you can't take 2 out of 0 so borrow 1 from 7."
> "9 minus 8 = 1 and 0 can't go into 2, so I borrow 1."

There were 69 pupils who used such "confused vocabulary" in the subtraction exercises.

The first multiplication exercise was arranged this way: 19 x 20 = ___. There were 148 pupils who first rewrote the exercise vertically, some placing the 19 on top, others placing the 20 on top. The partial products were derived and written in many different ways. Here are some examples of the written work.

20	19	20	19	19	19	19	19	19
19	20	19	20	20	20	20	20	20
180	00	180	00	3800	19	218	00	20
20	38	200	380		28	00	218	
380	380	380	380		299	218	2180	

Other examples appear in the later analysis of wrong answers.

There were only 7 pupils who, without rewriting the factors, wrote the product 380, thinking simply '2 x 19 = 38 and add 0.'

A strategy used in finding the product 58 / 75 has been illustrated in earlier notes on addition strategies. It involved deriving an unknown combination from a known one. Usually this strategy was used with the product 7 x 8. A pupil would say "8 x 5 = 40, + 8 is 48, + 8 is 56," or he would say "7 x 7 = 49; + 7 = 56." There were 45 pupils who relied on this strategy in this particular exercise.

In the exercise: multiply 304 / 506 there were 101 pupils who arranged the partial products this way 1824 / 000 / 1520. Thirty-six others used a variety of other forms illustrated as follows.

1824	1824	1824	1824	1824
152000	0000	15200	0000	1520
	15240000		1520000	
1824		1824		
0000		0000		
152000		15200		

In the division exercises there was little evidence in the vocabulary used by the pupils of thinking of division as the inverse of multiplication. For example, in the exercise 27/81 there was very little thinking in the pattern 27 x ? = 81. Much more frequently the thinking was expressed in such words as: "27 divided by 81"; "27 goes into 81"; "How many 27's in 81"; "put 27 into 81"; "27 into 81"; there were 155 pupils who used such a pattern of thinking in the division exercises.

To find a digit in the quotient 29 pupils added the divisor aside. Thus for the exercise 27/81 a pupil might add like this, or he would estimate the quotient to be 4 and check by adding a column of four 27's.

$$\begin{array}{r} 27 \\ \underline{27} \\ 54 \\ \underline{27} \\ 81 \end{array}$$

A quotient digit was more often (79) chosen quite arbitrarily--usually too small--and checked by multiplication, aside. If the first product did not seem suitable, another digit was tried and another product derived. In the exercise 74/6484, for example, the divisor might be multiplied separately by 3, 4, 5, 6, 7, 8, and 9 before deciding that the first quotient digit should be 8. There were 29 pupils who chose a quotient digit and checked by repeated multiplication, aside. There were 123 pupils who, with at least one of the division exercises, estimated the quotient digit in some fashion and then tested their estimate by multiplication, aside. As will be seen in the reporting of wrong answers later, these estimates were made in a variety of ways.

Strategies Frequently Used in Four Operations with Fractions

Among the four exercises in addition of fractions, all were arranged horizontally like this: 3/4 + 5/2 = ___, 3/8 + 7/8 = ___, 5 7/8 + 2 1/2 = ___, 2/3 + 1/2 = ___. There were 90 pupils who first rewrote one or more of these vertically before attempting to find a sum. In the case of 5 7/8 + 2 1/2, there were 98 who added the whole numbers and fractions separately while 17 first changed the mixed numbers to improper fractions and then wrote with common denominators. As would be expected, most of the pupils (97) who tried the addition of fractions, first rewrote them as equivalent fractions with common denominators; 20 of these kept the horizontal form of the exercise. The incorrect practice of adding numerators for the numerator of the sum and the same with denominators was used by 62 pupils. A smaller number (10) added numerators and placed the sum over the larger denominator of the fraction addends, as in 3/4 + 5/2 = 8/4, and 6 pupils added numerators but multiplied denominators as in 3/4 + 5/2 = 8/8.

The subtraction exercises with fractions numbered five. Three were arranged horizontally like this: 3/4 - 1/2 = ___, 5/8 - 1/3 = ___, 7 1/2 - 4 1/4 = ___. The last of these was used in only four schools. The other two subtraction exercises were arranged vertically like this: $\begin{array}{r} 8\ 2/5 \\ \underline{4\ 3/10} \end{array}$, $\begin{array}{r} 9\ 2/3 \\ \underline{5\ 7/8} \end{array}$. The latter was used in only two schools. Again many pupils (94) first rewrote the horizontal exercises vertically. The whole numbers and fractions of mixed numbers were subtracted separately by 120 pupils while 4 first changed to improper fractions. There were 98 who converted to equivalent fractions with common denominators before subtracting. The incorrect practice of subtracting numerators for the numerator of the difference and the same for denominators, as in 5/8 - 1/3 = 4/5, was used by 49 pupils. The exercise 9 2/3 - 5 7/8 was usually rewritten as 9 16/24 - 5 21/24 = ___, then pupils "borrowed 1 from 9" and 38 of them correctly rewrote the 1 as 24/24 and added to 16/24 for 40/24, but 12 pupils "added the 1 to 16" by making it 26 for 26/24.

All three of the multiplication exercises with fractions were arranged horizontally like this: 2/3 x 3/5 = ___, 2 1/2 x 6 = ___, 5 1/2 x 3/4 = ___. There were 85 pupils who correctly multiplied numerators for the numerator of the product and the same for the denominators; two pupils first wrote equivalent fractions as in 2/3 x 3/5 = 10/15 x 9/15 = 90/225. Mixed numbers were rewritten as improper fractions first, as in 2 1/2 x 6 = 5/2 x 6/1, by 38 pupils. In this same exercise, however, 29 pupils multiplied the whole numbers and simply affixed the fraction for a product, as in 2 1/2 x 6 = 12 1/2, and 7 pupils found the product this way 2 1/2 x 6 = (2 x 6) + (1/2 x 6) = 12 + 3 = 15.

In the product of a mixed number by a fraction, 32 pupils rewrote the mixed number as an improper fraction first as in 5 1/2 x 3/4 = 11/2 x 3/4 = 33/8, while 30 pupils incorrectly multiplied the fractions and affixed the whole number for a product, as in 5 1/2 x 3/4 = 5 3/8. Another incorrect practice of writing equivalent fractions with a common denominator and then multiplying the numerators written over the common denominator for a product, as in 2/3 x 3/5 = 10/15 x 9/15 = 90/15, was used by 31 pupils. Interestingly there were 5 pupils who wrote the reciprocal of the second factor and multiplied the resulting fractions as in 2/3 x 3/5 = 2/3 x 5/3 = 10/9.

The four division exercises were all written horizontally, like this: 9/10 ÷ 3/10 = ___, 15 3/4 ÷ 3/4 = ___, 6 9/10 ÷ 3 = ___, 7/8 ÷ 2/3 = ___. There were 36 pupils who correctly wrote the reciprocal of the divisor and multiplied as in 7/8 ÷ 2/3 = 7/8 x 3/2 = 21/16, and there were 5 pupils who multiplied numerators and denominators without first writing the reciprocal of the divisor, as in 7/8 ÷ 2/3 = 14/24. The most frequent (69 pupils) incorrect practice was to divide numerators and place the quotient over the common denominator, as in 9/10 ÷ 3/10 = 3/10. There were 46 pupils who wrote mixed numbers as improper fractions before dividing, as in 15 3/4 ÷ 3/4 = ___, 63/4 ÷ 3/4 = ___ and 36 pupils who wrote equivalent fractions with a common denominator and divided numerators, as in 7/8 ÷ 2/3 = 21/24 ÷ 16/24 = 1 5/16. Many, who followed this practice, made the final division incorrectly and wrote a quotient of 1 5/24. In dividing a mixed number by a fraction, 46 pupils incorrectly divided the fractions and affixed the whole number for the answer, as in 15 3/4 ÷ 3/4 = 15 1/1 = 16. In dividing a mixed number by a whole number 26 pupils divided the whole numbers and affixed the fraction, as in 6 9/10 ÷ 3 = 6 3/10, while 3 pupils correctly divided the whole number and then the fraction, separately, by the whole number, as in 6 9/10 ÷ 3 = 2 + 3/10 = 2 3/10.

If a pupil followed the rather conventional strategies described here, he might have "cancelled" as indicated.

$$2/\underset{1}{\cancel{3}} \times \overset{1}{\cancel{3}}/5 = 2/5 \quad : \quad 2\ 1/2 \times 6 = 5/\underset{1}{\cancel{2}} \times \overset{3}{\cancel{6}}/1 = 15$$

$$9/10 \div 3/10 = \overset{3}{\cancel{9}}/\underset{1}{\cancel{10}} \times \overset{1}{\cancel{10}}/\underset{1}{\cancel{3}} = 3$$

$$15\ 3/4 \div 3/4 = \overset{21}{\cancel{63}}/\underset{1}{\cancel{4}} \times \overset{1}{\cancel{4}}/\underset{1}{\cancel{3}} = 21$$

$$6\ 9/10 \div 3 = \overset{23}{\cancel{69}}/10 \times 1/\cancel{3} = 23/10$$

Only 33 pupils used such cancellation strategy and 30 of these were in two schools. It must be remembered that many pupils did the multiplication and division exercises incorrectly and, therefore, never wrote them in a form where cancellation was useful. Even among the successful pupils, such practices as the following were followed.

2/3 x 3/5 = 6/15 = 2/5

15 3/4 ÷ 3/4 = (15 x 4/3) + (3/4 x 4/3) = 60/3 + 12/12 =
240/12 + 12/12 = 252/12 = 21

2 1/2 x 6 = 5/2 x 6/1 = 30/2

6 9/10 ÷ 3 = 69/10 · 1/3 = 69/30

15 3/4 ÷ 3/4 = 63/4 x 4/3 = 252/12

Nature of Wrong Answers--Whole Numbers

There were 2173 possible answers to the 13 exercises with whole numbers by the 170 pupils interviewed (one exercise was used in only 2 schools with 61 pupils). Of these possible answers 1658 (76%) were right; 449 (21%) were wrong; and 66 (3%) were omitted. Each wrong answer and a detailed account of how it was derived appears in Appendix A. It is hoped that the reader will read many of the explanations of how these wrong answers were derived for then it will become apparent how widely they vary--both numerically and in method of derivation. This variation again makes it clear how impossible it is to classify these methods of derivation into categories without destroying the idiosyncracies. Nevertheless the following general observations are offered with the hope that the reader will form his own list.

Addition

1. Many combinations were recalled incorrectly in addition, as in "9 + 8 = 18, + 1 = 19 and 8 is 27," or "7 x 8 = 63," or "8 x 5 = 35," or "7 x 8 = 48." The same was true in each of the other operations.

2. When counting was used pupils often lost count of the counting as in counting 9 on to 17 in 17 + 9 and getting 25, or in "7 x 8 = 57 (7 x 5 = 35; 7 x 6 = 42; 7 x 7 = 49: 50, 51, 52, 53, 54, 55, 57, 57").

3. Many pupils failed to add the carried digit even when it was written above the top digit in the column to the left.

4. Sometimes the wrong digit was carried from the sum of one column to the next as in "2 + 3 = 5, + 9 = 14, + 7 = 21, put down the 2 and carry 1."

5. Intending to add, a pupil may have, in fact, multiplied as in "7 + 2 is 14, + 5 is 19, + 2 is 21, + 4 is 25."

Subtraction

6. Some pupils intended to subtract but in fact divided as in 93 - 32, "2 from 3 is 1; 3 from 9 is 3."

7. A wrong order was often used in subtraction as in 86 - 49, a pupil would say "9 minus 6 is 3; 8 - 4 is 4" or "9 from 6 leaves 3 and 4 from 8 leaves 4" or in 708 - 329 a pupil said "9 from 8 is 1 ; 2 from 0 is 0; 7 from 3 is 4."

8. A pupil would think to borrow to increase a digit but not reduce the digit from which borrowed, as in 86 - 49 "16 - 9 = 7 (counting); 8 - 4 = 4."

9. When borrowing, as in 708 - 329, a pupil might borrow twice, once to make 0 a 10, and again to make 8 an 18, leaving the 7 as 5.

10. Some pupils borrowed from the tens column only, when they should have borrowed from both tens and hundreds columns, as in 708 - 329, rewrote 708 as 7-9-18, then "18 from 9 is 9; 9 from 2 is 7; 7 from 4 is 3."

11. Other pupils borrowed from the hundreds column only and rewrote the tens digit incorrectly. As in 708 - 329; rewritten as 6-10-18. Then 18 - 9 = 9; 10 - 2 = 8; 6 - 3 = 3.

12. The minuend was rewritten simply by affixing ones where needed, as in 708 - 329 which became 7-10-18 for 708 and the answer was 18 - 9 = 9; 10 - 2 = 8; 7 - 3 = 4.

Multiplication

13. The ones digit was multiplied by the ones digit and the tens digit by the tens digit only, as in 19 x 20 = ___, rewritten as 19 over 20, then "0 x 9 = 0 and 2 x 1 = 2" answer 20; or 58 x75 "5 x 8 = 40; 7 x 5 = 35, + 4 = 39. Written as a single product 390.

```
 19          58
 20         x75
```

14. The carried number was not included in the partial product, as in 19 x 20 = ___, rewritten as 19 over 20, then "0 x 9 = 0; 0 x 1 = 0; 2 x 9 = 18; 2 x 1 = 2."

Throughout the remaining pages of this report a blank space or blank spaces after a numeral indicates an indentation in the arrangement of a partial product. For example, partial products 1824, 000, and 1520- were arranged by the pupil this way

```
  1824
   000
 1520
 17024
```

or partial products 1824, 000-, 1520-- were arranged this way.

```
   1824
   000
 1520
 153824
```

15. Place value of partial products was confused as in 19
"0 x 9 = 0; 0 x 1 = 0; 2 x 9 = 18; 20
2 x 1 = 2, +1 = 3." 3800
Or in 304 "6 x 304 = 1824; 0 x 304 = 000; 5 x 304 = 1520-
506
for sum 17024.

16. The wrong product was written when one factor was 0, as in 19 x 20 = ___; "0 x 9 = 9; 0 x 1 = 1; 2 x 9 = 18; 2 x 1 = 2, +1 = 3." Pupil wrote 38- under 19 for sum of 399.

17. A multiplication fact was recalled incorrectly as 7 x 8 = 54, in 58 x 75 "8 x 5 = 40; 5 x 5 = 25, + 4 = 29; 7 x 8 = 54; 7 x 5 = 35, + 4 = 40." Then 290 + 404- = 4330.

18. One of the digits in the multiplier was not used in finding the product, as in 304; only two partial products 6 x 304 =
x506
1824; 5 x 304 = 1520-. Sum 17024.

19. Partial products were found correctly but errors were made in adding them, as in 58 5 x 58 = 290; 7 x 58 = 406- for sum
x75
4360, said "9 + 6 = 16" in adding partial products.

Division

20. A remainder was interpreted wrongly as in 27/$\overline{81}$; 81 ÷ 27 = 3; 3 x 27 = 81; 81 - 81 = 0; "27 won't go into 0, so answer is 30"; or in 48/$\overline{93}$ "48 goes into 93 one time; 1 x 48 = 48; 93 - 48 = 45; 48 can't go into 45; put 0 up; 45 - 0 = 45" for answer 10 R45.

21. "Long division" was confused with "short division" as in 27/$\overline{81}$; "2 goes into 8 four times; 2 x 4 = 8. Then 81 - 8- = 01; 2 won't go into 1" so answer is 4 R1. Or in 48/$\overline{93}$ "4 goes into 9 two times; 4 x 2 = 8; 9 - 8 = 1; bring down 3; 8 goes into 13 one time; 8 x 1 = 8; 13 - 8 = 5" answer 21 R5.

22. Quotient digit was multiplied by the divisor incorrectly, as in 74/$\overline{6484}$; "8 x 74 = 572" (8 x 4 = 32; 8 x 7 = 54, + 3 = 57).

23. Errors were made in repeated multiplications to find quotient digit, as in 74/$\overline{6484}$ decided 74 goes into 648 seven times, then 7 x 74 = 658 (thought 7 x 4 = 28 and 7 x 7 = 56, + 7 = 63).

24. Derived an answer before operation was complete, as in 74/$\overline{6484}$, "74 goes into 648, eight times; 648 - 592 = 56," so answer is 8 R56.

25. By repeated multiplication tried incorrectly to derive entire quotient instead of one digit at a time, as in 74/6484, multiplied 74 by 12, by 24, by 52 and by 61. Chose 52 for quotient "because 3848 is closest to 6484": then 6484 - 3848 = 2636. Placed 2734 (incorrect product of 74 x 61) under 2636. Then 2636 - 2734 = 102. Answer 5261 R102.

26. Place value was handled incorrectly in the quotient, as in 15/7590; "15 goes into 75 five times 75 - 75 = 0; bring down your 9; 15 won't go into 9 so bring down 0; 6 x 15 = 90" so answer is 56. Or 15 into 75 five times: 75 - 75 = 0 "15 won't go into 0 so bring down 9; 15 won't go into 9 so bring down 0; 15 into 90 goes 6 times: 90 - 90 = 0; 15 into 0 zero times" so answer is 560.

Nature of Wrong Answers--Fractions

There were 2640 possible answers to the 16 exercises in computation with fractions by the 176 pupils interviewed. (One exercise was used in only four schools with 115 pupils and another in only two schools with 61 pupils.) Of these possible answers, 924 (35%) were right; 865 (33%) were wrong: and 851 (32%) were omitted. Each of the wrong answers and a detailed account of how it was derived appears in Appendix B. First it should be observed that the performance with fractions was much below that with whole numbers. There was a lower percent of right answers, a higher percent of wrong answers and a higher percent of omissions. The higher percent of omissions is explained, in part, by time limitations which meant that some of the later exercises in the total set could not be completed in the time devoted to an interview.

Again the reader will observe great variations in the answers, and their derivations, for the fraction exercises. There were, however, many more common wrong answers, and like derivations, for fractions than for whole numbers. Some observations are offered below regarding the wrong answers and their derivations with the fraction exercises. Again the reader is urged to examine carefully Appendix B to get a clearer idea of the nature and derivation of these answers.

Addition

1. A prevalent practice was to add numerators and place the sum over one of the denominators or over a common denominator, as in 3/4 + 5/2 = 8/4 "5 + 3 = 8. You don't add the bottom numbers because 2 will go into 4."

2. The most prevalent practice in adding fractions was to add numerators for the numerator of the sum and the same for the denominators, as in 3/4 + 5/2 = 8/6 or 3/8 + 7/8 = 10/16.

3. Many errors were made as pupils undertook to write equivalent fractions with common denominators, as in 3/4 + 5/2 = ___: chose 4 as C.D. Then, for 3/4, "4 times 1 equals 4 and 1 + 3 is 4," so 4/4; for 5/2 "2 x 2 = 4 and 4 x 5 = 20," so 20/4; or 8/8 for 7/8 ("8 into 8 one time and 1 + 7 = 8"). The same thing was done in the other operations.

4. Several relatively large whole number answers were a surprise, as in 3/4 + 5/2 = 86 (5 + 3 = 8; 4 + 2 = 6) or 3/4 + 5/2 = 59 ("4 and 5 is 9; 3 and 2 is 5"), or 3/8 + 7/8 = 26 ("7 over 8 is 15; 3 over 8 is 11; 15 + 11 = 26").

5. The numerator and denominator of one fraction were added for the numerator of the sum, and the same with the second fraction for the denominator of the sum, as in 3/8 + 7/8 = 11/15 ("8 and 3 is 11; 7 and 8 is 15"), or 3/4 + 5/2 = 7/7 ("3 + 4 = 7; 2 + 5 = 7"), or 2/3 + 1/2 = 5/3 ("2 + 3 = 5; 2+ 1 = 3").

Subtraction

6. As in addition, a very prevalent practice was to subtract numerators for the numerator of the difference and the same with denominators; as in 3/4 - 1/2 = 2/2 (3 - 1 = 2; 4 - 2 = 2); or 8 2/5 - 4 3/10 = 4 1/5 (8 - 4 = 4; 3 - 2 = 1; 5 from 10 is 5), or 7 1/2 - 4 1/4 = 3 0/2 (7 - 4 = 3; 1 - 1 = 0; 2 from 4 = 2), or 5/8 - 1/3 = 4/5 (1 from 5 is 4; 3 from 8 is 5).

7. In writing equivalent fractions, some pupils divided a denominator into the C.D. and added this quotient to the numerator of the original fraction for the numerator of the equivalent fraction, as in 3/4 = 4/4 (4 goes into 4 one time; 3 + 1 = 4). Others subtracted for the new numerator, as in 5/8 = 2/24 ("8 goes into 24, three times, 3 take away 5 is 2").

8. As in addition, some surprising whole numbers were derived for answers, as in 3/4 - 1/2 = 22 ("2 take away 4 is 2; 1 take away 3 is 2"), or 8 2/5 - 4 3/10 = 394 ("2 over 5 would leave 3; 3 over 10 would leave 9; 4 from 8 would leave 4"), or 7 1/2 - 4 1/4 = 133 ("1 over 2 leave 1; 1 over 4 would be 3; 7 from 4 would leave 3").

9. There were cases of the wrong use of borrowing, as in 8 2/5 - 4 3/10 = 3 9/5 (borrowed 1 from 8; made it a 7; changed 2 of 2/5 into 12; then 7 12/5 - 4 3/10 = 3 9/5), or 8 2/5 - 4 3/10 = 3 1/10 (wrote 4/10 for 2/5 and 4/10 for 3/10; "you can't subtract 4 from 4, so you borrow 1 from 4 [remainder from 8 - 4] make it a 3." Made first 4/10 into 5/10, then 5/10 - 4/10 = 1/10.

10. A frequent error in writing equivalent fractions was to choose a C.D.; use it for the denominator of the new fraction but retain the numerator of the old fraction; as in 5/8 = 5/24 and 1/3 = 1/24.

11. The borrowed number was used incorrectly as in 9 2/3 - 5 7/8 rewritten as 9 16/24 - 5 21/24. Then 8 26/24 - 5 21/24.

Multiplication

12. Many pupils first wrote equivalent fractions, unnecessarily, and then incorrectly multiplied numerators and placed the product over the C.D., as in 2/3 x 3/5 = 10/15 x 9/15 = 90/15, or 2/3 = 7/15 ("3 goes into 15 five times; 5 + 2 = 7") and 3/5 = 6/15 ("5 goes into 15 three times; 3 + 3 = 6"). Then 7/15 x 6/15 = 46/15 because 6 x 7 = 46, or 2 1/2 x 6 = 5/2 x 12/2 = 60/2.

13. Here, as in addition and subtraction, surprisingly large whole numbers were derived as products, as in 2/3 x 3/5 = 100 ("2 x 5 = 10, put down 0 and carry 1; 3 x 3 = 9, + 1 = 10. Answer 100"), or 2 1/2 x 6 = 120 (wrote vertically with 6 below 2 1/2. Then "0 times 1/2 = 0; there is nothing under 1/2 so multiply by 0; 6 x 2 = 12, answer 120"), or 2/3 x 3/5 = 615 ("2 x 3 = 6; 3 x 5 = 15").

14. In all the operations there were examples of correctly derived answers with errors introduced with conversions to simpler form, as in 2 1/2 x 6 = 5/2 x 6/1 = 30/2 = 15/2 ("2 goes into 30 fifteen times, and the denominator is 2"), or 2/3 x 3/5 = 2/3 ("2/3 x 3/5 = 6/15, to reduce divide by 3/3; 6 goes into 3 two times; 15 goes into 3 three times, so that'll be 2/3").

15. Some pupils wrote the reciprocal of the second factor before multiplying, as in 2/3 x 3/5 = 2/3 x 5/3 = 10/9, or 2 1/2 x 6 = 5/2 x 1/6 = 5/12.

16. In a mixed number times a fraction the fractions would be multiplied and the whole number affixed, as in 5 1/2 x 3/4 = 5 3/8 ("1 x 3 = 3; 2 x 4 = 8; bring over 5"), or in 5 1/2 x 3/4 = 5 3/2; 5 1/2 = 5 2/4 and 5 2/4 x 3/4 = 5 6/4 = 5 3/2.

17. In a mixed number times a whole number the whole numbers would be multiplied and the fraction affixed, as in 2 1/2 x 6 = 12 1/2 ("6 x 2 = 12, bring over 1/2").

Division

18. As in multiplication a widely used practice was to divide numerators and place the product over the C.D., as in 9/10 ÷ 3/10 = 3/10, or even in 7/8 ÷ 2/3 = 3/2 ("2 goes into 7 three times; 3 goes into 8 two times"), or 15 3/4 ÷ 3/4 = 63/4 ÷ 3/4 = 21/4.

19. After writing equivalent fractions errors were made in dividing numerators, as in 7/8 ÷ 2/3 = 21/24 ÷ 16/24 = 1 5/24, or 21/24 ÷ 16/24 = 1 R5.

20. In dividing a mixed number by a whole number the whole number was divided by the whole number and the fraction was affixed, as in 6 9/10 ÷ 3 = 2 9/10.

21. In dividing a mixed number by a fraction, the fractions were divided and the whole number affixed, as in 15 3/4 ÷ 3/4 = 15 1/4 ("3 ÷ 3 = 1, bring over 15, the answer is 15 1/4"), or 15 3/4 ÷ 3/4 = 16 ("3/4 ÷ 3/4 = 1, bring over 15 and 15 + 1 = 16").

22. Numerators of like fractions were multiplied instead of divided, as in 9/10 ÷ 3/10 = 27/10 ("the denominator would be 10; 3 x 9 = 27, and 27 would be numerator"), or in 15 3/4 ÷ 3/4 = 15 9/4 ("bring over 15; 3 x 3 = 9; bring over 4").

23. Numerators and denominators were multiplied without writing a reciprocal of the divisor, as in 6 9/10 ÷ 3 = 69/10 x 3/1 = 207/10.

Reasons Given for Wrong Answers to Comparison Exercises with Fractions

There were 1408 possible answers to the eight comparison exercises with fractions. Of these, 498 (35%) were right answers; 239 (17%) were wrong answers; and 671 (48%) were omitted. Here, as in the other fractions exercises, the large number of omissions resulted, for the most part, from the limitation of time which meant that some pupils did not get to try the final exercises during the time devoted to the interviews. It may be more significant to note that of the 737 answers to exercises attempted 68% were right and 32% were wrong. Moreover, it is true that many right answers were chosen for wrong reasons or because the wrong reasons happened to produce a correct choice in one instance and a wrong response in another. An example of this may be seen in the following pupil's choices and reasons.

Which is larger?

✓ 2/3 × 5, or	___ 1 × 5	"2/3 is greater than 1"
✓ 3/2 × 6, or	___ 1 × 6	"3/2 is greater than 1"
✓ 17 ÷ 5/8, or	___ 17 ÷ 1	"5/8 is greater than 1"
✓ 17 ÷ 5/2, or	___ 17 ÷ 1	"5/2 is greater than 1"
✓ 3 9/10 + 7/8, or	___ 3 9/10 + 1	"7/8 is greater than 1"
✓ 8 5/6 + 7/4, or	___ 8 5/6 + 1	"7/4 is greater than 1"
✓ 10 1/9 - 7/8, or	___ 10 1/9 - 1	"7/8 is greater than 1"
✓ 12 3/8 - 5/4, or	___ 12 3/8 - 1	"5/4 is greater than 1"

Another example is another pupil's correct choice for a faulty reason.

Which is larger ___ 3 9/10 + 7/8 or ✓ 3 9/10 + 1

"3 9/10 + 7/8 = $3\,\frac{9+7}{10+8}$ = 3 16/18 and 3 9/10 + 1 would be 4"

Still another example appears in this pupil's two responses.

Which is larger ✓ 17 ÷ 5/8 or ___ 17 ÷ 1 ?
"17 ÷ 5/8, you come out with a mixed number"
Which is larger ✓ 17 ÷ 5/2 or ___ 17 ÷ 1 ?
"17 ÷ 5/2, you come out with a mixed number."

There were really two comparisons necessary in these exercises. First the pupil needed to decide whether the fraction was larger or smaller than 1. Then he must decide the effect on the operation of the relative value of the fraction and 1. For example in the exercise: which is larger ___ 17 ÷ 5/8 or ___ 17 ÷ 1 ? , a correct choice could involve the decision first that 5/8 is less than 1 and then the decision that dividing a constant by a smaller number produces a larger quotient. Among the wrong answers, some pupils made the first of these decisions incorrectly; others made the wrong second decision; still others made both decisions wrongly. Some examples of these and other reasons for wrong answers follow.

1. The fraction and the 1 were incorrectly compared. For example a pupil may decide that all fractions--proper and improper--are less than 1, or that they are all greater than 1, as in the pupil responses above, or 12 3/8 - 5/4 is greater than 12 3/8 - 1. "5/4 is less than 1; if you subtract less than 1, you get more than if you subtract more than 1."

2. Whichever divisor was thought to be larger, this made the corresponding sum, difference, product, or quotient larger. For example this pupil had these two incorrect responses.

17 ÷ 1 is larger than 17 ÷ 5/8 "1 is larger than 7/8"
17 ÷ 5/2 is larger than 17 ÷ 1 "5/2 is larger than 1"
Another pupil said 10 1/9 - 1 is larger than 10 1/9 - 7/8 because "1 is a whole and 7/8 is not a whole" or 10 1/9 - 1 is larger than 10 1/9 - 7/8. "10 1/9 is on both sides: change 1 to a fraction of 8/8. Then 8/8 is greater than 7/8."

3. Many pupils attempted to perform the operations before making the comparison. Often these were carried only far enough to enable the pupil to make a choice. Often these operations were incorrect and led to incorrect choices. The interviewer tried to discourage this method of answering the questions, but many pupils seemed to understand nothing else and persisted. Here is an example.
17 ÷ 1 is larger than 17 ÷ 5/8 because "5 x 17 = 85; 85 ÷ 8 = ? ; 8 into 8 goes 1 time; 8 into 5 goes no times;" hence 10 is less than 17.
Or this example:
2/3 x 5 is larger than 1 x 5. "Cause 1 x 5 is 5, and 3 x 5 is 15 and if I make it out a fraction it will be 2 x 15 which is greater than 5."
Or, 17 ÷ 1 is larger than 17 ÷ 5/8 "17 ÷ 1 = 17; 17 ÷ 5/8 goes 3 2/8"
Or, 2/3 x 5 is larger than 1 x 5. "1 x 5 = 5; 2/3 x 5 = 5 2/3, which is larger than 5."

4. Some pupils who performed the operations--correctly or incorrectly--compared the sums, differences, products or quotients incorrectly, as in 8 5/6 + 1 is greater than 8 5/6 + 7/4. "8 5/6 + 1 = 9 5/6 and 8 5/6 + 7/4 would only be 8 12/6."
Or in 1 x 6 is greater than 3/2 x 6. "1 x 6 is 6 and 3/2 x 6 would be 18/2; then 18/2 is less than 6."
Or in this unusual one, 17 ÷ 1 is greater than 17 ÷ 5/8 "you have to take 5 into 17 and 8 into 17, and you only take that (17 ÷ 1) 1 time."
Or in 3 9/10 + 7/8 is larger than 3 9/10 + 1 "3 9/10 + 7/8 = 3 16/18 and 3 9/10 + 1 = 4 9/10; then 3 16/18 is larger than 4 9/10."

5. A common error was to think of all whole numbers as greater than fractions, as in 8 5/6 + 1 is greater than 8 5/6 + 7/4 "1 is a whole number and 7/4 is a fractional number."
Or in 10 1/9 - 1 is greater than 10 1/9 - 7/8 "this is a whole and that's just 7/8" or "1 is a whole and 7/8 is half of a whole."

Some Characteristics of Good and Poor Computers

One of the assumptions of this study was that "there are observable differences in the patterns of thinking--computational strategies of successful computers and unsuccessful computers." To help in testing this assumption a selection was first made of 12 pupils from among the 176 interviewed. Six of these composed a "good computer" group--one from each of the six schools. The pupil chosen from each school was the one who did most, or all, the exercises and had the largest number of correct answers. The remaining six pupils composed the "poor computer" group--one from each of the six schools. Again the pupil chosen from each school was one who had tried most, or all, the exercises and who had the largest number of incorrect answers. These were not the poorest computers in their several schools for most of the very poor computers were not able to complete most of the exercises. Verbatim transcriptions of the interviews with these 12 pupils were made and carefully examined to detect features in the computational practices of the two groups.

Two lists of features were thus prepared. Then tapes of interviews with other good and poor computers were heard again. This time they were heard particularly for the purpose of detecting contrasts in the computational practices of good and poor computers. The lists which were developed in this fashion follow.

A. Good Computers

1. Good computers know the basic combinations and do not need to derive them by primitive methods such as counting.

2. Good computers tend to follow conventional algorithms rather consistently. They remember what they have been taught to do and follow the orthodoxy of classroom and textbook quite closely. For example, in column addition they are more likely, than are poor computers, to add the digits in order from top to bottom, or bottom to top, rather than to jump about to find preferred combinations such as doubles or sums of ten. A sentence arranged horizontally such as 3/4 + 5/2 = ___ is as likely to be rearranged vertically before rewriting with C.D. by good computers as by poor ones.

3. Good computers use pencil and paper more than would appear necessary--especially with simple exercises. They do, however, tend to do more "mental arithmetic" than do poor computers. For example a girl with an I.Q. of 121, who did all the exercises with only 3 errors, first tried the exercise

19 x 20 = ___ by saying "9 x 0 is 0; 0 x 1 is 0"; she hesitated and asked "can I write this in column form?" When told "yes" she rewrote the 20 under 19, then said--quite comfortably now--"0 times 9 is 0; 0 time 1 is 0; 2 times 9 is 18; write 8 and carry 1; 2 x 1 is 2, + 1 is 3." The work was arranged like this. With partial products written, she then said: "Add, bring down 0; 8 + 0 = 8; bring down 3, answer is 380."

```
  19
  20
  00
 38
 380
```

Another example is the bright pupil who found the product 2/3 x 3/5 = ___ this way. $\frac{2}{3} = \frac{10}{15}$ 10/15 x 9/15 = 90/225 = 2/5.

3/5 = 9/15

4. Good computers appear to be much less dependent on the arrangement of an exercise--vertical or horizontal (especially noted in fractions)--to provide a clue to the appropriate algorithm than do poorer computers. For example, two exercises in fractions were 3/4 - 1/2 = ___ and

```
 8 2/5
-4 3/10
```

The good computer was quite likely to rewrite both as equivalent fractions and subtract correctly. A poor computer might have done one of these one way and the other another way as did this pupil.

3/4 - 1/2 = 2/4 ("3 - 1 = 2; 2 can go into 4 so I can use 4 for the determinant").

```
8 2/5  = 4/10
4 3/10 = 3/10
4        1/10
```

5. The good computers did much better than the poor ones in the final group of eight comparison exercises both with their choices and their reasons. The answers of a good computer will illustrate. 3 9/10 + 1 is larger than 3 9/10 + 7/8 "because 1 is greater than 7/8." 8 5/6 + 7/4 is greater than 8 5/6 + 1 "because 7/4 is 1 3/4. You are only adding 1 here; and you are adding 3/4 extra here." 10 1/9 - 7/8 is greater than 10 1/9 - 1 "because you are only subtracting 7/8 here and here you are subtracting a whole, like 8/8." 12 3/8 - 1 is greater than 12 3/8 - 5/4 "because 5/4 is greater than 1." 1 x 5 is greater than 2/3 x 5 "because 1 is greater than 2/3." 3/2 x 6 is greater than 1 x 6 "because 3/2 is 1 1/2." 17 ÷ 5/8 is greater than 17 ÷ 1 "because 5/8 is less than 1; so you are dividing by more here" (17 ÷ 1). 17 ÷ 1 is greater than 17 ÷ 5/2 "because this is like 2 1/2 (5/2) and if you divide 17 by 2 1/2 you won't get as great a number as 17."

6. Good computers seem to have better memories. For example, once they have identified an exercise as requiring a certain algorithm, they are quite likely to remember and use it correctly. In an exercise such as 7/8 ÷ 2/3 the good computer decides the appropriate rule is "write the reciprocal of the divisor and multiply." He remembers the rule and uses it correctly. The poor computer often has difficulty in deciding whether this rule is used in multiplication or division and whether it is the divisor or the dividend that is written as a reciprocal.

7. Good computers, more often than poor computers, appeared to sense when an answer was wrong and proceeded to make corrections. For example one pupil first got 315 as a quotient for 15/7590. In multiplying the 5 of 315 by 15 he discovered that 5 x 15 = 75. He readily saw that this meant the 3 of 315 was wrong. He did the exercise over and produced the correct answer 506.

8. The thinking of good computers often ran ahead of their words or pencils. One pupil in adding the column 7 + 5 + 2 + 4 with a carried 2 above the 7 said "that's 2 and 4 is 8, + 5 is 13, + 7 = 20." Her thinking actually combined the two 2's first and then 4 + 4 = 8, + 5 = 13, + 7 = 20. Another pupil in adding 64
78
said "8 and 4 is 12, carry my 1, 6 and 3 is 13 and 1 makes 14." She explained that she said "6 and 3" instead of "6 and 7" because she was thinking of the 3 of 13.

9. Good computers tried out computations mentally and quickly as in finding a common denominator or a quotient digit. For example, a pupil quickly chose 24 as the C.D. of 5/8 and 1/3 after quickly trying "in her mind" 12 ("3 will go into 12 but 8 won't"), then 16 ("8 will go into 16 but 3 won't"); then 18 ("3 would go into 18 but 8 won't"); finally 8 x 3 = 24. Another pupil in choosing the first quotient digit for 74/6484 said "7 goes in 64 nine times, but 9 x 4 is 36 which would make it 66, so you multiply 8 x 74." Many other pupils laboriously made the multiplications aside with pencil and paper.

B. Poor Computers

1. Poor computers' stock of whole number facts is limited. They rely heavily on a few retained facts such as doubles, or products with 5 as one factor, from which to derive unknown combinations. They make extensive use of counting to make combinations.

2. Poor computers often make errors in whole number operations when their counting, or other derivations of unknown combinations, become too involved for their short memory spans. For example, a pupil may try to derive a combination such as 7 x 9 by recalling 7 x 7 = 49; then adding by counting 49 + 7 = 56 and 56 + 7 but getting something other than 63.

3. Poor computers have much more trouble with fractions than with whole numbers. Primitive methods such as counting are not as useful in fractions as in whole numbers, although a few pupils tried to think with "pieces of pie" in operating with fractions.

4. Poor computers have difficulty remembering the conventional operational algorithms--especially in fractions. Moreover they have difficulty in matching those they do remember with the right exercise. So they devise simple, and what seems to them as obvious, procedures such as adding numerators and then adding denominators for the sum of two simple fractions.

5. When poor computers encounter difficulty with an improvised algorithm, they often switch to something else that will produce an answer, however remote from the proper procedure it may be. For example, in the exercise 7/8 ÷ 2/3 = ___ one pupil said "2 divided into 7; you can't do that; so it would be ... (pause); change this (7/8) to 8/7", wrote 8/7 ÷ 2/3 "2 divided by 8 is 4; 3 divided by 7 is ..." (pause); "it would have to be 21 (3 x 7)."

6. Poor computers tend to rely more on aids to memory. For example, in adding the column 9 + 8 + 1 + 8 (9 at the top) one pupil thought "8 + 8 = 16, + 1 = 17" but then wrote aside 17 and said "9 + 7 = 16 (counting dots); put down my 6

$$\frac{9}{26}$$

and carry my 1; 1 and 1 are 2."

7. What appear to be careless errors of poor computers are often supported by a reason--even if faulty. For example, in multiplying 304 by 6 a pupil wrote 1804, seemingly failing to add the carried 2 to the 0 of 6 x 0 = 0. Actually the pupil said "6 times 4 is 24; put down the 4 and carry 2 (written above 0 of 304); 0 times 2 = 0; 6 times 3 = 18."

8. Poor computers have great difficulty with long division as in $74\overline{)6484}$. They will make several trial multiplications aside in an effort to find a quotient digit. The factors they choose to multiply the divisor by are often quite arbitrarily chosen. In the exercise $48\overline{)93}$ a pupil multiplied

aside 48 by 2; by 3, by 5, by 7, by 8, and by 9 in an attempt to find the correct quotient digit. In long division, other poor computers will design unorthodox algorithms that yield incorrect answers. An example is the practice of dividing by one digit of the divisor at a time.

9. Poor computers often did not hesitate to reverse minuend and subtrahend in subtraction. For example, 86 - 49 may yield 43 as an answer with this explanation "6 from 9 is 3; 4 from 8 is 4"; or in the incorrect solution of 8 2/5 - 4 3/10 = 4 1/5 one pupil said "8 - 4 is 4; 3 - 2 is 1; 5 - 10 is 5."

10. Poor computers frequently confuse 0 and 1, as in 15 3/4 ÷ 3/4 = 15 0/4 or in rewriting 708 as 7-9-18. This pupil said "subtract 1 from 0 and leave it 9; make 8 into 18." In another exercise 19 x 20 = ___ another pupil wrote 19 above 20 and said "9 x 0 = 0; 0 x 1 = 1" for a partial product of 10. Then 9 x 2 = 18; 2 x 1 = 2, + 1 = 3" for partial product of 38-.

11. Poor computers are quite likely to be confused in arranging the partial products of multiplication--especially when the factors contain zeros. For example, in 304 x506 a pupil wrote

```
 1824
 1520
-----
17024
```

There were many more examples of other faulty arrangements of partial products in this exercise.

Conclusions

For the 176 pupils interviewed in this study it appears that these conclusions are justified.

1. Pupils did vary widely in the computational strategies they employed in exercises with whole numbers and with fractions.

2. Some orthodox strategies were used infrequently. For example, few pupils in the division of fractions wrote the reciprocal of the divisor and multiplied. Unorthodox strategies were frequently observed--some yielding correct answers and some incorrect ones.

3. There was very little evidence of "mental computation," that is, independence of pencil and paper--even in such simple exercises as 9/10 ÷ 3/10 = ___ or 3/4 - 1/2 = ___.

4. There was heavy dependence on the arrangement of the exercise to provide a clue to computational strategy. Many pupils proceeded comfortably with an operation only after they had written in vertical form an exercise presented horizontally in sentence form.

5. Early developmental strategies were often retained. This seemed to be more often the case in operations with whole numbers than with fractions. The frequent reliance on counting in operations with whole numbers is an example.

6. The vocabulary often did not correctly express a pupil's thinking. For example, in 86 - 49, a pupil would say "you can't take 6 from 9" or "9 into 6 won't go" so borrow 1 and make 6 a 16. Then 9 from 16 is 7."

7. Good computers recalled basic combinations readily and followed orthodox strategies closely. Poor computers often derived basic combinations they could not recall and devised quite unorthodox strategies to do this.

8. Very little practice of testing, by estimate, the reasonableness of answers was observed.

9. It was especially apparent in long division that many pupils depended upon written trial and error to find quotient digits.

10. Some of the concepts emphasized in "Modern Mathematics" programs were infrequently apparent in the computational strategies employed. For example, few pupils thought of "regrouping" the 8 tens and 6 ones in 86 as 7 tens and 16 ones. They much more frequently thought "borrow one from 8 and add it to 6." Moreover, few thought of subtraction as the inverse of addition or division as the inverse of multiplication.

11. In the fraction comparison exercises, where pupils were asked to use their pencils only to check one of two choices, many needed actually to perform the indicated operation before making a choice.

12. There was a frequent practice of continuing a computational strategy throughout the exercises of one group. The same was true from one group to another if the exercises were similarly arranged. For example a pupil who added numerator and denominators in 3/4 + 5/2 = ___ would continue to do so with the remaining addition exercises with fractions. Or a pupil who subtracted the smaller digit from the larger in 86 - 49 = 43 would continue to do so in 708 - 329 to get 421.

13. Recorded interviews with individual pupils is a promising technique for identifying computational strategies of pupils. It is promising for research as well as the classroom teacher.

Recommendations

1. Recorded interviews with pupils as they compute should be more widely used in research studies. They should also be used by the practicing teacher to the extent that time permits. It is suggested that early each fall a teacher do three to five interviews in each class taught. Pupils should be selected for interviews on the basis of their written work. For the interview, a variety of simple exercises should be prepared covering computational skills which have been studied previously. The guides for interviews suggested earlier in this study should be observed. It is expected that these sample interviews will give a teacher much clearer insights into the backgrounds of his pupils than would otherwise be likely. Moreover, these insights are likely to be helpful in planning further work with the pupils interviewed as well as the others in the class.

2. Supervisors should also learn the technique of interviewing and use it to prepare taped recordings for use in in-service meetings with teachers.

3. Some school systems have experimented with the employment of what have been called "diagnostic-prescriptive teachers." These full-time special teachers work individually with pupils referred to them by regular classroom teacher. When the pupil goes back to his regular class, the "diagnostic-prescriptive teacher" sends to his teacher a diagnosis of his learning difficulties and some suggestions for correcting them. This appears to be a promising practice. In the hands of such a helping, special teacher the recorded interview, such as used in this study, should be very helpful. For example, a pupil's teacher may be advised whether the peculiar computational strategy he uses should be refined and improved or replaced because it is wrong arithmetically or is too awkward.

4. Teachers should encourage pupils to reveal individual strategies in their day-to-day computational exercises. They should not conceal their individual strategies through fear of ridicule. Rather there should be recognition for originality in thinking.

5. Teachers should give much more attention to teaching pupils to check the reasonableness of answers.

6. Bright pupils should be encouraged to develop independence of pencil and paper in many computations. They should not be "impeded" in their quick mental reactions by a requirement to "show your work."

APPENDIX A

Wrong Answers--Whole Numbers

73 + 24 = ___.

Answers

79 "3 plus 4 = 7 and 7 plus 2 = 7,8,9," wrote 7 in tens and 9 in ones column.
107 "4 plus 3 = 7 and 2 plus 8 = 10".
98 "4 and 3 is 8; 2 and 7 is 9".
87 "4 plus 3 is 7 and 7 plus 2 is 8".
98 "3 and 4 is 8; 7 and 2 is 9".
107 "4 + 3 = 7; 7 + 2 is 10".
102 Misread 73 as 76 then thought (76 - 1) + (24 + 1) = 75 + 25 = 100. Somehow recorded 102.
96 "7 + 2 = 9 ; 3 + 4 = 6.

Add 64
78

Answers

132 Failed to add the carried one in ten's column.
141 "8 = 4 is 11 ; 7 + 6 + 1 = 14".
132 "8 + 4 = 12 ; 7 + 6 = 6 + 6 + 1 = 13".
115 "6 + 7 = 13 put down 1 carry 3 (above 4) 8 + 4 + 3 = 15.
143 "4 + 8 = 13, 6 + 7 + 1 = 14".
152 "8 + 4 = 12 ; 8 + 6 = 14, + 1 = 15".
132 "8 + 4 = 12 ; 7 + 6 = 13 (counted fingers).

Add 709
538
291
478

Answers

2015 "8 and 1 is 8, plus 8 = 16, plus 9 = 25".
2017 "9 + 8 = 18, plus 1 = 19, plus 8 = 27." Counted by moving pencil point in pairs.
2017 "9 and 8 is 18, one more is 19, and 8 is 27".
2014 "8 + 1 = 9, plus 8 = 15, plus 9 (counting fingers) = 24".
2024 "8 + 1 = 9 ; 9 + 8 = 19, plus 9 = 24" carried 2, "3 + 2 = 5, plus 9 = 14, plus 7 = 22" (counted for sums).
2017 "9 and 8 is 17 ; 17 + 8 = 26, and 1 = 27.
1819 "9 + 8 = 17, + 1 = 18, + 9 = 29" for ones column. "2 + 7 = 9, + 5 = 14, + 2 = 16 and 16 + 4 = 18" for hundreds column.
1916 Correct sum of 26 for ones column and 21 for tens column but carried 1 instead of 2 to hundreds column.
1916 "9 + 1 = 10 ; 10 + 8 = 18 ; 18 + 8 = 25 (miscounted fingers) tens column correct. +7 + 2 = 8, plus 2 = 10, plus 5 = 15, + 4 = 19" for hundreds column.

Add 799
538
281
478 (continued)

1945 "9 + 8 = 17, + 1 = 18, + 8 = 25" for ones column, carried 2. "0 + 3 = 3, + 9 = 12, + 7 = 14" for tens column, carried 1. "7 + 5 = 13, + 2 = 14, + 4 = 18, + 1 = 19' for hundreds column.

1916 ones and tens columns correct, "4 + 2 = 6, + 5 = 11, + 7 (counting) = 17, + 2 (counting) = 19" for hundreds column.

2015 "8 + 1 = 9, + 8 = 17 ; 17 + 9 (counting by twos) = 25".

2026 "9 + 3 = 13, + 7 = 20, + 2 = 22" for tens column.

2017 "9 + 8 = 17, + 1 = 18 + 8 (counting fingers) 19, 20, 21 . . . 27".

2026 ones and hundreds columns correct. For tens column started with 7 and counted up the column, stopping at 22 instead of 21.

2015 For ones column, "9 + 8 = 17, 8 + 1 = 9, and 17 + 9 = 25" to add 17 + 9 said "7 and 9 is 16 ; 1 plus 1 is 2".

2015 "8 and 8 is 16, + 1 is 17, and 9 is 25" (counting fingers).

3014 "8 and 1 is 9 : and 9 is 18 ; 18 and 8 is 34" for ones column. "5 + 2 = 7, and 7 is 14 ; 14 + 4 is 28 (wrote aside for 28), + 2 is 30" for hundreds column.

14
4
--

2017 "8 + 8 = 16, + 1 is 17 ; 17 and 9 is 27" (counting) for ones column.

2414 "9 and 8 is 17, and 1 is 18, and 8 is 24" for ones column. "7 and 5 is 12, and 2 to carry is 14, and 2 is 16, and 4 is 24" for hundreds column.

2014 "9 + 1 = 10, + 8 = 18, + 8 (counting) = 24".

2018 "9 and 8 are 17, and 1 is 18, + 8 (counting in pairs) = 28".

18,216 Correct 26 for ones column, correct 21 for tens column. Wrote 21 and carried 2 to hundreds column. 18 for hundreds column--without carried 2--wrote 18 by 216.

2026 "7 + 3 = 10, + 9 is 19, + 2 is 22" for tens column.

2018 "9 and 1 is 10, and 8 is 18, and 8 more (counting) is 28" for ones column.

2516 "7 + 2 (carried) is 14, + 5 is 19, + 2 is 21, + 4 is 25 for hundreds column.

2014 "8 + 1 is 9, + 8 is 17, + 9 (counting) is 24" for ones column.

2017 "9 and 8 is 17, and 1 is 19, and 8 is 27." (For 19 + 8 thought 19 + 1 is 20, + 7 is 27) for ones column.

2046 "2 (carried) + 3 = 5, + 9 = 14, + 7 = 24 for tens column (for 14 + 7, thought 7 x 3 = 24).

2014 "9 + (8 + 1) = 18, + 8 (counting) = 24" for ones column.

2017 "9 + 8 (counting fingers) = 17, + 8 (counting fingers) = 26, +1 = 27" for ones column.

2019 "10 + 8 is 18, and 8 is 29" (for 18 + 8, thought 8 + 9 = 19, and 1 + 1 = 2).

2015 "9 + 8 = 17, + 1 = 18, + 8 = 18, 19, 20 . . .25" (counted fingers).

Add 792
538
291
473
(continued)

2014 "9 + 8 is 15, + 1 = 16, and 8 more would be 24".
1227 Started on left, got 17 for hundreds column. Wrote 1 and carried 7 above 10's column; got 23 for sum of tens column, wrote 2 carried 3 to one's column. Then 3 + 9 = 16, + 9 = 24, + 3 = 26, + 1 = 27. Wrote 7 in one's column of sum.
2046 One's and hundreds columns correct. For ten's column 2 (carried) + 0 = 2, + 3 = 5, + 9 = 14, + 7 = 24. Explained "14 + 6 = 20 plus 4 left over = 24".
1907 "9 and 8 is 17, and 1 is 19 and 18 is 27". "0 and 3 is 3, and 9 is 12, plus 7 is 19, and 1 is 20". "7 and 5 is 12, plus 2 is 14, and 4 is 18, and 1 is 19".
1986 1st column correct. Then 9 + 7 = 16, + 2 = 18; skipped the addend "3". Hundreds column correct with wrong carried 1.
2014 "9 + 1 = 10, + 8 = 18, + 9 = 24 (counted fingers).

93 - 32 = 1

Answers

31 "2 from 3 is 1 ; 3 from 9 is 3".
31 "3 minus 2 is 1 ; 3 from 9 is 3".
31 "3 from 2 = 1 ; 3 from 9 = 3".
31 "3 from 2 is 1, and 9 from 3 is 3".
51 "2 from 3 is 1 ; and 3 from 9 is 5".
31 "3 from 9 is 3 (thinking 3 x 3) and 2 from 3 is 1"
71 "2 from 3 is 1, and 9 - 3 is 7".
31 "3 - 2 = 1 ; and 3 - 9 = 3 (thinking: 3 goes into 9 three times).
151 Vertically "13 - 2 = 11 ; carry 1. That would make 18, 18 - 3 = 15".

Subtract 86
49

Answers

43 "9 from 6 = 3 and 8 from 4 = 4".
43 "9 - 6 = 3 ; 8 - 4 = 4".
40 "9 from 6 is 0 ; 4 from 8 is 4".
44 "8 - 4 = 4 and 6 - 9 is 4".
43 "9 - 6 = 3 ; 8 - 4 = 4".
43 "9 from 6 is 3 ; 8 from 4 is 4".
47 "9 from 6 is 7 ; 4 from 8 is 4".
35 "16 take away 9 is 5 ; 7 take away 4 is 3".
43 "9 from 6 is 3 ; 4 from 8 is 4".
117 Borrow 1 from 8, make 6 a 16. Counted 9, 10, 11 ... 16 "that's 7". Then "7, 8, 9, 10, 11" that's 117.
43 "9 take away 6 is 3 ; 8 take away 4 is 4".

Subtract 86
49 (continued)

35 "9 from 16 is 5, and 4 from 7 is 3".
36 "9 from 16 is 6 (counting 9 to 16), 4 from 7 is 3".
43 counted "7, 8, 9" wrote 3 in ones place, "4 from 8 is 4"
39 "9 from 16 is 9 ; 4 from 7 is 3".
43 "9 - 6 = 3 and 8 - 4 = 4".
35 "9 from 6 is 5, and 4 from 7 is 3".
43 "9 from 6 leaves 3 and 4 from 8 leaves 4".
37 "16 take away 9 (counting 9, 10, 11 ... 16) is 5 ; 7 take away 4 is 3".
38 "16 - 9 (counting, 9, 10, 11 ... 16) = 8 : 7 - 4 = 3".
35 Wrote 7 for 8 and 16 for 6. Said "that's 5" for 16 - 9 and "that's 3" for 7 - 3.
35 "16 - 9 = 5 and 7 - 4 = 3".
35 "6 and 9 is 15" ; 7 - 4 is 3".
43 Counted "6, 7, 8, 9", wrote 3. Then "4 from 8 is 4".
36 "9 from 16 is 6 ; 4 from 7 is 3".
47 "16 - 9 = 7 (counting) ; 4 - 8 = 4".
35 "9 from 16 = 5 ; 4 from 7 = 3".
38 "16 - 9 = 8 ; 7 - 4 = 3".
43 "6 subtract from 9 will be 3 ; 8 subtract 4 will be 4".
36 "16 - 9 = 6 ; 7 - 4 = 3".
27 "16 - 9 = 7 ; 7 - 4 = 2".
38 "9 from 16 is 8 ; 4 from 7 is 3".

Subtract 708
329

Answers

279 "You have to borrow from 7 ; make it a 6. Then borrow from 6 to make 0 a 10. Borrow from 10 to make 8 an 18. Then 18 - 9 = 9 ; 9 - 2 = 7 ; and 5 - 3 = 2".
289 Borrow from 7, make it a 6 ; make 8 an 18, then 18 - 9 = 9. can't take 2 from 9, so borrow from 6, make it a 5 and 0 a 10. 10 - 2 = 8 and 5 - 2 = 3. (7 pupils)
479 Borrowed from 0 made it a 9 and 8 an 18. Then 18 - 9 = 9; 9 - 2 = 7 and 7 - 3 = 4. (10 pupils)
401 "9 from 8 is 1 ; 2 from 0 is 0 ; 7 from 3 is 4".
389 Borrowed from 7, made 8 an 18 and 0 a 10. Then 18 - 9 = 9 ; 10 - 2 = 8 ; 6 - 3 = 3. (4 pupils)
401 "9 - 8 = 1 ; 0 - 2 = 0 ; 7 - 3 = 4".
400 "9 won't go into 8" so wrote 0, "2 won't go into 7" so wrote 0, "3 goes into 7, four times".
369 Wrote 6 - 9 - 18 for 708, then 18 - 9 = 9 ; 9 - 2 = 6, and 6 - 3 = 3.
421 "9 from 8 is 1 ; 2 subtract 0 is 2 ; 7 - 3 = 4".
421 "8 from 9 is 1 ; 0 from 2 is 2 ; 3 from 7 = 4".
139 "9 from 18 is 9 ; 2 from 5 is 3 ; 3 from 4 = 1".

Subtract 708
329 (continued)

459 "18 - 9 = 9 ; 7 - 2 = 5 ; 7 - 3 = 4".
381 "9 from 8 is 1 borrow 1 from 7, make it a 6 and 0 a 10. Then 2 from 10 = 8 and 6 from 3 is 3".
479 "Borrow 1 from 0, make it a 9 and 8 an 18. Then half of 18 is 9 ; 2 from 9 is 7, and 3 from 7 is 4".
369 Wrote 6 - 9 - 18 for 708. Then "9 and 9 is 18; 2 from 9 is 6 (counting), 3 from 6 is 3".
401 "8 from 9 is 1 ; 0 from 2 is 0 ; 3 from 7 (counting 3, 4, 5, 6, 7) is 4".
381 "9 - 8 = 1 and 0 can't go into 2 so I borrow from 7, make it a 6 and 0 a 10." Then "10 - 2 = 8 and 3 - 6 = 3".
389 "9 from 18 is 9 ; 2 from 9 is 8 ; 3 from 6 is 3".
401 "8 from 9 is 1 ; 0 from 2 is 0 ; 3 from 7 is (3, 4, 5, 6, 7) 4".
309 "Borrow from 7 leaves 6 put 1 at 8 and 18 from 9 is 9, you can't take 2 from 0, so it's 0, and 6 from 3 is 3".
388 "Borrow from 7, make it 6, make 0 a ten, change 8 to 9." Then "9 + 3 = 17 ; 17 - 9 = 8 ; 10 - 2 = 8 ; 6 - 3 = 3".
489 Change 8 to 18 ; 18 - 9 = 9. "Can't take 0 from 2 so change 0 to a 10 ; 10 - 2 = 8 ; 7 - 3 = 4". (2 pupils)
377 Wrote 708 as 6 - 9 - 18. Then 18 - 9 = 7 (thinking 8 + 9) : 9 - 2 = 7 ; 6 - 3 = 3.
401 "9 take away 8 leaves 1 ; 2 subtract 0 leaves 0 ; 7 take away 3 leaves 4".

19 x 20 = ___

Answers*

20** "0 x 9 = 0 ; 2 x 1 = 2". (8 pupils)
20 "9 x 0 = 0 ; 1 x 2 = 20".
390 "9 x 0 = 0 ; 0 x 1 = 1" so 10 for 1st partial product. "9 x 2 = 18 ; 2 x 1 = 2, + 1 = 3, so 38- for 2nd partial product.
3800 "0 x 9 = 0 ; 0 x 1 = 0 ; 9 x 2 = 18 ; 2 x 1 = 2, + 1 = 3." Arranged vertically. Wrote single product 3800. (3 pupils)
480 "9 x 0 = 0 ; 9 x 2 = 18 ; 1 x 0 = 0 ; 1 x 2 = 3" 180 + 30- = 480.
2180 "9 x 0 = 0; 9 x 2 = 18." Wrote 80 and carried 1, then "1 x 20 = 20, + 1 = 21." Wrote 21 with 80 for 2180.

---

* A blank indicates a partial product was indented. For example, partial products of 10 and 38 were written 10.
38

** It will help in reading these accounts if this product is first rewritten, as the pupils did, either as 19 over 20 or 20 over 19.

19 x 20 = ___ (continued)

299 "0 x 9 = 9 ; 0 x 1 = 1 ; 2 x 9 = 18 ; 2 x 1 = 2". Wrote 19 + 28- for sum of 299.

389 "0 x 9 = 9 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 2 x 1 = 2, + 1 = 3". Wrote 09 and 38- for sum of 389.

399 "9 x 0 = 9 ; 9 x 2 = 18 ; 1 x 0 = 1 ; 1 x 2 = 2". Wrote 189 and 210 for sum 399.

200 "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 2 x 1 = 2". Wrote 00 and 28-. In adding "0 ; 0 from 8 is 0 ; 2" for 200.

2180 "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 2 x 1 = 2". Wrote single answer 2180.

20 "1 x 0 = 0 ; 1 x 2 = 2".

20 "0 from 19 is 0 ; and 2 from 1 is 2".

280 "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 2 x 1 = 2". Added 00 + 28- for 280. (6 pupils)

38 "9 x 2 = 18 ; 2 x 1 = 2, + 1 = 3".

180 "9 x 0 = 0 ; 9 x 2 = 18 ; 1 x 0 = 0 ; and 1 times 0 again". Wrote 180 and 00- for sum of 180.

399 "0 x 9 = 9 ; 0 x 1 = 1 ; 2 x 9 = 18 ; 2 x 1 = 2, + 1 = 3". Wrote 38 under 19 for sum 399.

48 "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 2 x 1 = 3, + 1 = 4". Wrote 00 + 48 = 48.

3800 "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 2 x 1 = 2, + 1 = 3". Wrote single product 3800.

57 "9 x 0 = 9 ; 0 x 1 = 1 ; 2 x 9 = 18 ; 2 x 1 = 2, and 1 = 3". Wrote 38 under 19 for sum of 57.

300 "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 2 x 1 = 2, + 1 = 3". Arranged 00 and 38-. Said "bring down 0 ; 0 + 8 = 0 ("can't add nothing and 8 and get 8"), 3.

360 "0 x 9 = 0 ; 9 x 2 = 16 ; 1 x 0 = 0 ; 1 x 2 = 2". Added 160 and 20- for 360.

218 "2 x 1 = 2 ; 2 x 9 = 18 for 218. 0 x 1 = 0 ; 0 x 9 = 00, 218 + -00 = 218".

2180 "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 2 x 1 = 2 for 218-, 00 + 218- = 2180 .

480 Wrote 20 under 19 ; "put down 0 ; 9 x 2 = 18". Wrote 8 and carried 1 above 1 of 19. 2 x 2 = 4. .

308 Added column of 19 twenties correctly for 380 but wrote 308 for sum.

38 "0 x 19 = 00 ; 2 x 19 = 38 ; 00 + 38 = 38".

1820 "0 x 1 = 0 ; 0 x 9 = 0 ; 2 x 9 = 18 ; 2 x 1 = 2. 00 + 182- = 1820.

20 "0 x 9 = 0 ; 2 x 1 = 2" Doubted this answer so wrote column of 19 twenties and added correctly.

380 and 20. Wrote 20 under 19. "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18; 2 x 1 = 2, + 1 = 3" correctly. Then "0 x 9 = 0 ; 2 x 1 = 2." Said this was "short way" and both answers were correct only did them in different ways.

240 "9 x 0 = 0 ; 2 x 9 = 18 ; 1 x 0 = 0 ; 1 x 2 = 6, 180 + 60 = 240".

19 x 20 = ___ (continued)

20180 "0 x 9 = 0 ; 0 x 1 = 0 ; 2 x 9 = 18 ; 20 x 1 = 20
00 + 2018- = 20180"

2180 "9 x 0 = 0 ; 1 x 0 = 0 ; 9 x 2 = 18 ; carry 1 ; 2 x 1 = 2.
00 + 218- = 2180"

Multiply 58
75

Answers

4450 "5 x 8 = 40 ; 5 x 5 = 35, + 4 = 39" Second product 406- sum of 390 + 406- = 4450

4330 "8 x 5 = 40 ; 5 x 5 = 25, + 4 = 29 ; 7 x 8 = 54 ; 7 x 5 = 35, + 5 = 40" Added 290 + 404- for 4330. (5 pupils)

4345 "8 x 5 = 35 ; 5 x 5 = 25, + 3 = 28" Second product 406- sum of 285 and 406- = 4345.

4450 290 for first product, then "7 x 8 = 56 ; 7 x 5 = 35, + 5 = 41" Sum of 290 + 416- = 4450.

4370 Correct partial products 290 and 406-. Added "0, 9 + 6 = 17 ; 2 + 1 = 3, and 4" for 4370.

4250 Correct partial products 290 and 406-. Added "0 ; 9 + 6 = 15 ; 2 + 0 = 2, and 4" for 4250.

4320 1st partial product 290. Then "7 x 8 = 63 ; 7 x 5 = 35, +6 = (counted) 40." Added 290 + 403- for 4320.

4550 1st partial product 490, second 406-, sum 4550

390 "5 x 8 = 40 ; 7 x 5 = 35, + 4 = 39" wrote single product 390. (8 pupils)

688 1st partial product 290, then "7 x 8 = 48 ; 7 x 5 = 35, + 4 = 39" Added 290 + 398 for 688.

4230 Correct partial products 290 and 406-. Then "0 from 0 is 0 ; 6 from 9 is 3 ; 0 from 2 is 2 ; 4 from nothing is 4" .

35820 "8 x 5 = 40 ; 5 x 5 = 25, + 4 = 28" ; "7 x 8 = 54" (wrote down 7 rows of 8 marks each and counted) ; 7 x 5 = 35. Second partial product written 3554-. Sum 280 + 3554- = 35820.

4370 1st partial product 290; then "7 x 8 = 57 (said "7 x 5 = 35 ; 7 x 6 = 42 ; 7 x 7 = 49 ; 7 x 8 = 49, 50, 51, 52, 53, 54, 55, 56, 57") 7 x 5 = 35, + 5 = 40" 290 + 407- = 4370.

4140 1st partial product 290 ; "7 x 8 = 35 ; 7 x 5 = 25, 26, 27 ... 35 (on fingers), + 3 = 38." Sum of 290 + 385- = 4140.

40890 1st partial product 290. "7 x 8 = 56 ; 7 x 5 = 35." Wrote 3 partial products 290, 56- and 35--, sum 40890.

6455 "5 x 8 = 45 ; 5 x 5 = 25, + 4 = 29 ; 7 x 8 = 56 ; 7 x 5 = 35, + 6 = 61" Wrote 295 as 1st partial product and 6160 as second.

4360 Correct partial products 290 and 406- in adding said "9 + 6 = 16".

Multiply 58 (continued)
75

1350 1st partial product 290. Then "7 x 8 = 56 (wrote 6 and carried 5) ; 7 x 5 = 35, that'll be one zero" wrote 106- as second partial product. 290 + 106- = 1350.

4690 1st partial product 290 ; "7 x 8 = 50 (6 x 6 = 36, + 7 = 43, + 7 = 50) ; 7 x 5 = 35, + 4 (carried from 5 x 8) = 39, + 5 (carried from 7 x 8 = 50) = 44." Then 290 + 440- = 4690.

4760 1st partial product 290. "7 x 8 = 57 (7 x 7 = 49 and 8 more, 50, 51, 52 ... 57) ; 7 x 5 = 35 (5, 10, 15 ... 35) + 4 = 39 (counted) + 5 = 44 (counted)." Then 290 + 447- = 4760.

975 "5 x 8 = 40 ; 5 x 5 = 25 ; 7 x 8 = 56 ; 7 x 5 = 35." Wrote 4 partial products 40, 25, 560, and 350. Sum 975.

3820 1st partial product 290; "7 x 8 = 63 (wrote 7, 14, 21 ... 63), 7 x 5 = 35". Wrote 353- as 2nd partial product. Then 290 + 353- = 3820.

4230 Correct partial products 290 and 406- then adding: "bring down 0 ; 6 and 9 = 3, then 2, and bring down 4".

4760 1st partial product 290; "7 x 8 = 57 (7 x 7 = 49 and 8 more 50, 51, 52 ... 57) ; 7 x 5 = 35 (5, 10, 15 ... 35), + 4 (36, 37 ... 39), + 5 = 40." Wrote 447- as 2nd partial product. Then 290 + 447- = 4760.

3350 1st partial product 290 ; then "7 x 8 = 56 ; 7 x 5 = 35, + 5 = 30" Wrote 306- as second partial product.

4450 1st product 290; then "7 x 8 = 56 (wrote 6 as carried digit) ; 7 x 5 = 35, + 6 = 41." Then 290 + 416- = 4450.

3760 "8 x 7 = 56 ; 7 x 5 = 35, + 2 = 37." Wrote 3760 as single product.

——— 1st product 290 ; then 7 x 8 = 63 (counted by 7's) ; 7 x 5 = 35 (counted by 5's), + 6 (counted) = 41 for partial product of 4130. With prompting, corrected before completing.

4310 "5 x 8 = 40; 5 x 5 = 25." Wrote 250 as 1st product. Then 250 + 406- = 4310.

688 290 first product ; "7 x 8 (8, 16, 24 ... 48) = 48 ; 7 x 5 = 35, + 4 = 39." Then 290 + 398 = 688.

4250 290 and 4060 for partial products. "9 + 6 = 15 ; 2 + 0 = 2 ; bring down 4" (failed to add carried 1 from 9 + 6).

4320 290 first product. "7 x 8 = 53 (8 x 8 = 64 so 8 x 7 = 53) 7 x 5 = 35, + 5 = 40." Then 290 + 403- = 4320.

4410 290 first product. "7 x 8 = 62 ; 7 x 5 = 35, + 6 = 41" 290 + 412- = 4410.

4398 "8 x 5 = 40 ; 5 x 5 = (5, 10, 15 ... 30) 30, + 4 (31, 32,33) = 33" So first product 330 and 330 + 406- = 4398. Carelessly wrote 8 for 0 as ones digit.

1101 Starting on left, "7 x 5 = 35, carry 3 ; 7 x 8 = 56, + 3 = 59" for 559 ; "5 x 5 = 25 ; 5 x 8 = 40, + 2 = 42" for 542. 559 + 542 = 1101.

Multiply 58
75 (continued)

4450 5 x 8 = 40 ; 5 x 5 = 25, + 4 = 29 ; 7 x 8 = 56 ; 7 x 5 = 35, + 5 = 41 ; 290 + 416- = 4450.

6360 5 x 8 = 40 ; 5 x 5 = 25, + 4 = 29 ; 7 x 8 = 57 ; 5 x 7 = 35, + 5 = 60 ; 290 + 607- = 6360.

4285 "5 x 8 = 45 ; 5 x 5 = 25, + 4 = 29 ; 7 x 8 = 49 (7 x 6 = 42, + 7 = 49) ; 7 x 5 = 35, + 4 = 39" 295 + 399- = 4285.

5780 5 x 8 = 40 (counted by 5's keeping count on fingers) ; 5 x 5 = 25, + 4 = 28 ; "7 x 8 is 16, + 7 = 23, + 7 = 30, + 7 = 37, + 7 = 44, + 7 = 51." Then 51 + 4 (written above 5 of 58 from 5 x 8) = 55. Then 280 + 55-- = 5780.

4360 5 x 8 = 40 ; 5 x 5 = 25, + 4 = 29 ; 7 x 8 = 57 (7 x 7 = 49, 50, 51, 52 ... 57) ; 7 x 5 = 35, + 5 = 40. Then 290 + 407- = 4360.

4150 5 x 8 = 40 ; 5 x 5 = 25, + 4 = 29 ; 7 x 8 = 56 ; 7 x 5 = 35, + 5 = 38 ; 290 + 386- = 4150.

4366 5 x 8 = 45 ; 5 x 5 = 25, + 4 = 29 ; 7 x 8 = 56 ; 7 x 5 = 35, + 5 = 40. Then 295 + 406- = 4355.

445 5 x 8 = 45 ; 7 x 5 = 40, + 4 = 44.

Multiply 304
506

Answers*

16,804 "6 x 4 = 24 ; 6 x 0 = 0 ; 0 x 2 (carried) = 0 ; 6 x 3 = 18" for 1804. "5 x 4 = 20 ; 5 x 0 = 0: 0 x 2 (carried) = 0 ; 5 x 3 = 15" for 1500-.

161,964 "6 x 4 = 24 ; 6 x 2 (carried) = 12 ; 6 x 3 = 18, + 1 = 19" for 1924. "0 x 4 = 4 ; 0 x 0 = 0 ; 0 x 3 = 3" for 304-. "5 x 4 = 20 ; 5 x 0 = 5, + 2 = 7 ; 5 x 3 = 15" for 1570- confused columns in sum "4 ; 4 + 2 = 6 ; 9 ; 7 + 3 + 1 = 11 ; 5 + 1 = 6 ; 1."

171,804 "6 x 4 = 24 ; 6 x 0 = 0 ; 6 x 3 = 18" for 1804. "0 x 4 = 0 ; 0 x 0 = 0 ; 0 x 3 = 0" for 000- "5 x 4 = 20; 5 x 0 = 0 ; 5 x 3 = 15, + 2 = 17" for 1700--.

29,024 "6 x 4 = 24 ; 6 x 0 = 0, + 2 = 2 ; 6 x 3 = 18" for 1824. "5 x 4 = 20 ; 5 x 0 = 0, + 2 = 2 ; 5 x 3 = 18" for 1820-. 1824 + 1820- = 29024 (failed to add carried 1 from 8 + 2).

161,924 "6 x 4 = 24 ; 6 x 2 (carried) = 12 ; 6 x 3 = 18, + 1 (carried) = 19" for 1924. "0 x 4 = 0 ; 0 x 0 = 0 ; 0 x 3 = 0" for 000-. "5 x 4 = 20 ; 5 x 2 (carried) = 10 ; 5 x 3 = 15, + 1 (carried) = 16" for 1600--.

1500 "6 x 4 = 24 ; 6 x 0 = 6, + 2 = 8 ; 6 x 3 = 18" for 1884. "0 x 4 = 0 ; 0 x 0 = 0 ; 0 x 3 = 0" for 000-. "5 x 4 = 20 ; 5 x 0 = 0 ; 5 x 3 = 15" for 1500-. Added only 000- and 1500- for 1500.

---

* Unless otherwise indicated partial products were added correctly.

Multiply 304
506 (continued)

17024 "6 x 304 = 1824" ; then 0 x 4 = 0 ; 0 x 0 = 0 ; 0 x 3 = 0. No product written. "5 x 4 = 20 ; 0 x 2 = 2 ; 5 x 3 = 15" for 1520-. Sum of 1824 and 1520- = 17024.

16,584 "6 x 4 = 24 ; 6 x 0 = 0, + 2 = 8 ; 6 x 3 = 18" for 1884. "0 x 4 = 0 ; 0 x 0 = 0 ; 0 x 3 = 0" for 000-. "5 x 4 = 20 ; 5 x 0 = 0, + 2 = 7 ; 5 x 3 = 15" for 1570-. 1884 + 000- + 1570- = 16584 (failed to add carried 1 from 8 + 7).

1524 "4 x 6 = 24, put down 4 and remember 2 (written above 0 of 304) ; 2 and 0 and 0 is 2 ; 3 x 5 = 15" for 1524.

17024 Partial products 1824 ; 000- ; and 1520-. Sum 17024. (7 pupils)

1504 "6 x 4 = 24 ; 2 (carried) x 0 = 0 ; 3 x 5 = 15" for 1504.

1524 "6 x 4 = 24 ; 6 x 0 = 0, + 2 (carried) = 2 ; 5 x 3 = 15" for 1524.

3324 6 x 304 = 1824 ; wrote 3 zeros ; then 5 x 4 = 20 ; 5 x 0 = 0 ; 5 x 3 = 15 for 1500. Then 1824 + 000 + 1500 = 3324.

16,024 "6 x 304 = 1824 ; 5 x 304 = 1520-." Sum 1824 + 1520- = 16024. Failed to multiply by 0 of 506.

18,824 6 x 304 = 1824 ; 0 x 304 = 0000 ; for 2nd product "bring down 2 zeros ; 5 x 4 = 20 ; 5 x 3 = 15, + 2 = 17" for 17000. Sum = 1824 + 0000 + 17000.

159,924 "6 x 4 = 24 ; 6 x 0 = 6, + 2 = 8 ; 6 x 3 = 18" for 1884. "0 x 4 = 4 ; 0 x 0 = 0 ; 0 x 3 = 3" for 304-. "5 x 4 = 20 ; 5 x 0 = 5 ; 5 x 3 = 15" for 1550--.

161,824 6 x 304 = 1824 and 0 x 304 = 000-. Then "5 x 4 = 20 ; 5 x 0 = 0 ; 5 x 3 = 15, + the 2 + the 20 is 16" for 1600-. Sum of 1824 + 000- + 1600- = 16824 (columns confused)

17021 6 x 304 = 1824 ; 5 x 304 = 1520- added incorrectly for 17021, failed to multiply 304 by 0.

171804 6 x 304 = 1824 ; 0 x 304 = 000- ; "5 x 4 = 20 ; 5 x 0 = 0 ; 5 x 3 = 15, + 2 (carried from 5 x 4) = 17" for 1700-- ; added "4 ; 2 + 0 = 0 ; 8 + 0 = 8 ; 1 + 0 = 1 ; 7 ; 1."

29224 6 x 304 = 1824 ; 0 x 304 = 000- ; "5 x 4 = 24 ; 5 x 0 = 0, + 2 = 2". Wrote 24-- as 3rd product ; 5 x 3 = 15 ; wrote 15--- as fourth product. 1824 + 000- + 24-- + 15--- = 29224.

18,824 6 x 304 = 1824 ; "0 ; 5 x 4 = 20 ; 5 x 3 = 15, + 2 = 17" for 1700-. Failed to multiply 0 of 304 by 5. (2 pupils)

156,864 "6 x 4 = 14 ; 6 x 0 = 0, + 1 = 6 ; 6 x 3 = 18" for 1864. 0 x 304 = 000- ; "5 x 4 = 20 ; 5 x 0 = 0, + 1 (from 14 of 6 x 4) = 5 ; 5 x 3 = 15" for 1550--.

Multiply 304
506 (continued)

151,824	"6 x 2 = 12 and 2 more is 24 ; 6 x 0 = 0, + 2 = 2 ; 6 x 3 = 18 for 1824 ; 0 x 304 = 000- ; 5 x 4 = 20 ; 5 x 0 = 0 ; 5 x 3 = 15 for 1500--.
151,824	6 x 304 = 1824 ; 0 x 304 = 000- ; "5 x 4 = 20 ; bring down 0 (from 304) ; 5 x 3 = 15" for 1500--.
1506	"6 x 4 = 26, put down my 6 and carry 2, that's 0 and 3 x 5 = 15" for single product 1506.
151824	6 x 304 = 1824 ; 0 x 304 = 000- ; 5 x 4 = 20 ; 5 x 0 = 0 ; 5 x 3 = 15 for 1500--.
17204	"4 x 6 = 24 ; 6 x 3 = 18, + 2 (carried from 24) = 20." Wrote 204, wrote 0 under 4 of 204. Then "4 x 5 = 20 ; 5 x 3 = 15, + 2 (carried) = 17" for 17000 and 204 + 17000 = 17204.
151,804	"4 x 6 = 24 ; 0 x 6 = 0 ; 3 x 6 = 18" for 1804. 0 x 304 = 000- ; "4 x 5 = 20 ; 0 x 5 = 0 ; 3 x 5 = 15 ; for 1500--.
153821	"4 x 6 = 21 ; 6 x 0 = 0, +2 = 2, 6 x 3 = 18" for 1821. Then 0 x 304 = 000- ; 5 x 304 = 1520 for 1520--.
1,521,824	6 x 304 = 1824 ; 0 x 304 = 0000- ; 5 x 304 = 1520---.
23666	"4 x 6 = 26 (4 x 4 = 16, + 4 = 20, +6 = 26) ; 6 x 2 = 12 ; 6 x 3 = 22 (3 x 5 = 15, 16, 17, 18 ... 21, + 1 = 22)" ; wrote 0 under 6 of 2226 ; 0 x 4 = 4 ; 0 x 2 = 2 ; 0 x 3 = 3, +1 = 4 ; then wrote 0 in ones column ; 5 x 4 = 20 ; 5 x 2 = 10, + 2 = 12 ; 5 x 3 = 15, + 1 = 16, + another 1 = 17. 12 12 304 506 2226 4240 17200 23666
156,864	6 x 304 = 1824 ; "0 x 4 = 4 ; 0 x 0 = 0 ; 0 x 3 = 3" for 304- ; 5 x 304 = 1520 for 1520--.
161,924	"6 x 4 = 24 ; 6 x 2 = 12 ; 6 x 3 = 18, + 1 = 19" for 1924 ; 0 x 304 = 000-- ; 5 x 4 = 20 ; 5 x 2 = 10 ; 5 x 3 = 15, + 1 = 16 for 1600--.
157024	6 x 304 = 1824 ; 0 x 304 = 0000 ; 5 x 304 = 15200. Wrong sum from confused alignment of columns.
17024	6 x 304 = 1824 ; 5 x 304 = 1520 for 1520--. Failed to multiply by 0 of 506. (6 pupils)
173,884	"4 x 6 = 24 ; 6 x 0 = 6, + 2 = 8 ; 6 x 3 = 18" for 1884 ; 0 x 304 = 000- ; "5 x 4 = 20 ; 5 x 0 = 0, bring down 2 ; 5 x 3 = 15, + 2 = 17" for 1720--.
154,484	"6 x 4 = 24 ; 6 x 0, + 6 = 8 ; 6 x 3 = 24" for 2484. "5 x 4 = 20 ; 5 x 3 = 15" for 152000.
152,824	Correct partial products. 1824, 000- and 1520--. In adding said 4 + 0 = 0 ; 2 + 0 = 2 ; 8 + 0 = 8; 2 + 1 = 2 ; 5 + nothing is 5 ; bring down 1.
1824	6 x 304 = 1824, 0 x 304 = 000- ; failed to multiply by 5 of 506.

Multiply 304
<u>506</u> (continued)

152,024	6 x 304 = 1824 ; 0 x 304 = 000- ; "5 x 4 = 20 ; 5 x 0 = 0 ; 5 x 3 = 15" for 1502--.
1524	"6 x 4 = 24 ; and 0, 0, 0 ; 5 x 3 = 15" wrote one product, 1524.
4482	"4 x 6 = 28 ; 6 x 2 = 18 ; 6 x 3 = 24, + 1 = 25" for 2588 ; "0 x 4 = 4 ; 0 x 2 = 2 ; 0 x 3 = 3" wrote 324. "5 x 4 = 20 ; 5 x 0 = 5, + 2 = 7 ; 5 x 3 = 15" for 1570.
18,824	6 x 304 = 1824 ; "5 x 4 = 20 ; 5 x 3 = 15, + 2 = 17" for 17000 failed to multiply by 0 of 506.
16824	6 x 304 = 1824 ; 0 x 304 = 0000; 5 x 4 = 20 ; 5 x 0 = 0 ; 5 x 3 = 15 for 15000.
205,718	2015 000 4218 205718 "5 x 4 = 20. Put down 0, carry 2 ; 5 x 0 = 0 ; put down 0 ; 5 x 3 = 15, put down 15". 0 x 3 = 0 ; 0 x 0 = 0 ; 0 x 4 = 0 ; "6 x 4 = 24 ; carry the 2 ; 6 x 0 = 0, + 2 = 2 ; 6 x 3 = 18, put down 18".
18524	"4 x 6 = 24, carry 2 ; 6 x 0 = 0 ; 2 x 6 = 12. Put down 2, carry 1." "5 x 3 = 18, + 1 = 25. 0 x 4 = 0 ; 0 x 0 = 0, plus another 0." Wrote 2 zeros ; 5 x 3 = 15, + 1 (above 3 from 6 x 2 = 12) = 16. Then 2524 + 1600- = 18524.
17,924	6 x 4 = 24 ; write 4 carry 2 ; 6 x 2 (carried) = 12 ; carry 1 ; 6 x 3 = 18, + 1 = 19 for 1924. 0 x 4 = 0 ; 0 x 0 = 0 ; 0 x 3 = 0 ; 5 x 4 = 20 ; 5 x 2 (carried) = 10 ; 5 x 3 = 15, + 1 = 16. Then 1924 + 000- + 1600- = 17924.
1524	"4 x 6 = 24." Wrote 4 carried 2 ; "bring down the 2" ; 5 x 3 = 15 for 1524.
25774	6 x 4 = 24, carry 2 ; 6 x 2 (carried) = 12, carry 1 ; 6 x 1 (carried) = 6 ; 0 x 304 = 000 ; from left 5 x 4 = 20, wrote 2 carry 0, 5 x 0 = 5 ; 5 x 3 = 15. Then 624 + 000- + 2515- = 25,774.
1524	6 x 4 = 24 ; carry 2 ; 0 x 0 = 0, + 2 = 2 ; 5 x 3 = 15.
1508	6 x 4 = 18 ; carry 1 ; 0 x 1 = 0 ; 5 x 3 = 15.

Divide 27/$\overline{81}$

<u>Answers</u>

30	81 ÷ 27 = 3 ; 3 x 27 = 81 ; 81 - 81 = 0 "27 won't go into 0" so answer is 30. (2 pupils)
4	"2" into 8 = 4 ; 2 x 4 = 8 ; 81 - 8- = 1 ; 7 won't go into 1" started to make answer 40. Decided this was too much. Stopped.
21	81 ÷ 27 = 2 ; 27 x 2 = 54 ; 81 - 54 = 26 (thought of 81 as 7 - 10, said 10 from 4 is 6 ; 7 from 5 is 2); "27 goes into 26 one time."

Divide 27/81 (continued)

4	"2 into 8 goes 4 ; 7 into 1 won't go." Couldn't complete.
17 R204	"7 x 1 = 7, so put 7 up; 2 x 8 = 16 (put 6 below 1 of 81 and 1 above 8 of 81) : 6 + 1 = 7 ; 8 + 6 = 14" (counting) for 147; "6 + 7 = 13" (wrote 3 under 7 of 147 and carried 1 above 4 of 147) ; "7 - 3 = 4 ; 4 - 3 = 1, - 1 (carried above 4) = 0 ; 3 take away 1 is 2". (Worked: 17 / 27/81 / +6 / 147 / -3 / 204)
19	"27 can't go into 81, because 81 is bigger than 27". "8 can go into 27". Put down 27 marks, took away 8 had 19 left.
21	"Round 27 to 30 ; 30 goes into 80, two times ; 2 x 27 = 54 ; 81 - 54 = 27 ; 27 goes into 27 one time". (2 pupils)
4 R-1	"2 goes into 8 four times ; 2 x 4 = 8 ; 81 - 8- = 01 ; 2 won't go into 1" so answer is 4 remainder 1. (2 pupils)
2	"27 goes into 81 two times because 2 x 27 = 54"; 81 - 54 = 27, then 54 + 27 = 81. "I get mixed up on division."
40 R-1	"2 will go into 8, four times ; 2 x 4 = 8" wrote 8 under 8 of 81 ; "8 - 8 = 0, there's nothing so bring down the one", wrote 01 ; "7 won't go into 1 so put 0 up, have remainder 1".
4	"2 goes into 8 four times ; 4 x 2 = 8 ; 81 - 8- = 01 ; I don't know what to do with the 7".
No answer	"2 will go into 8 four times but 7 won't go into 1"; several completely irrational attempts, gave up.
40 1/2	"2 into 8 goes 4 ; put up 4 and 8 - 8 = 0 ; bring down 1 ; 7 won't go into 1, so put 0 up and 0 under 1". Then 1 - 0 = 1. This gave R 1/2. Could write R-1 but teacher says write as 1/2.
2 R31	"27 goes into 81 two times," put 2 in quotient and under 1 of 81, then 81 + 2 = 83 ; 27 x 2 = 54 ; 54 + 83 = 137 ; 54 + 54 = 108 ; 137 - 108 = 31.
15 1/27	"7 goes into 8 one time with 1 left over ; 2 goes into 11 (left over 1 and 1 of 81) 5 times with 1/27 left over".
11 R3	Multiplied 7 x 12 and 7 x 11. Decided 7 goes into 81 eleven times. Then 81 - 77 = 4 "bring down the 2(from 27)" to make 24. "7 goes into 24 three times with remainder 3."
40	"2 into 8 is 4 ; 4 x 2 = 8" ; 81 - 8- = 01 ; "27 won't go into 1, so 0".
3 R211	27 into 81, three times (thought "20, 40, 60, 80, that's 3 times"). 3 x 2 = 6 ; 3 x 7 = 21 for 621. Wrote 621 under 81. Then "1 - 0 = 1 ; 1 subtracted by 2 = 1 ; 6 - 8 = 2".
40 R1	"2 into 8 goes 4 times" 81 - 8- = 1 ; "7 can't go into 1 so it's 0 ; 0 x 7 = 0 with 1 R." (two pupils)

Divide 27/$\overline{81}$ (continued)

3 R10	"27 goes into 81 about 3 times" ; "3 x 7 = 21, carry 2, 3 x 2 = 5, + 2 = 7." 81 - 71 = 10.
25	27 into 81 goes 5 ; put above 1 of 81, 81 below 81 ; 21 into 81 goes 2 put 2 above 8 ; added 81 and 81 = 162.

Divide 48/$\overline{93}$

Answers

10 R45	"48 goes into 93 one time." 93 - 48 = 45 ; "48 goes into 45 zero times". 45 - 0 = 45.
1 R51	"48 into 93 = 1" mentally 2 x 48 = 96 and 96 - 93 = 3. Then 48 + 3 = 51.
20 R13	"8 goes into 3 zero times ; 4 into 8, two times ; 4 x 2 = 8 ; 8 x 0 = 0" ; then 93 - 80 = 13.
1 R41	48 into 93 goes 1 time ; then to get remainder thought "2 between 48 and 50 ; 40 between 50 and 90 ; there will be 1 left so remainder 41". The 1 left came from thinking 2 from 93 for 48 to 50. Then 50 to 90 = 40 and 1 left = 41.
1 R44	Tried 2 decided too large. Then 1 for quotient. Then 93 - 48 = 44 said "8 subtracted by 3 = 4".
13 R1	Tried 4 x 48, "too much," then 2 x 48, "too much, " then 3 x 48, "too much." Finally "8 into 9 goes 1 time". 93 - 8- = 13. Then "4 into 13 goes 3." 13 - 12 = 1, so 13 R1.
2	"4 into 9 goes 2 with 1 left over." Didn't know what to do with the 1. "Could put it over the 3, but that won't help because 8 doesn't go into 4"; gave up.
1 R25	Decided 48 goes into 93 one time. Placed 1 in quotient and 48 below 93. Then "8 from 13 is 5." Placed 5 below 8 of 48. Then "4 goes into 8 (from 8_{93}) 2 times". Wrote 2 by 5 for remainder of 25.
57 R186	"48 into 93 is 7" wrote 7 in quotient above 3 of 93. Wrote 93 below 93. Then "48 into 9 is 5." Put 5 in quotient above 9 of 93. Then added 93 + 93 = 186. Said "answer is 186, other answer is 57".
1	Decided that 48 goes into 93 one time ; placed 48 under 93. Could not proceed further.
10 R47	Decided there is one 48 in 93. Then 93 - 48 = 47. "48 won't go into 47 so put up 0".
1 R2	"48 goes into 93 one time" and 93 - 48 = 45. Wrote 93 / 48 / $\overline{45}$. Then "5 from 8 and get 3 ; 4 - 4 = 0". Wrote 3 under 45. Then 5 - 3 = 2. Answer 1 R2.
23 R1	"4 won't go into 9 but will go into 8 two times." 4 x 2 = 8 and 9 - 8 = 1. "4 won't go into 1 so bring down 3. 4 won't go into 13 but will go into 12 three times." 4 x 3 = 12 and 13 - 12 = 1.

Divide 48/$\overline{93}$ (continued)

5	Wrote down 48 marks. Counted off nines. Came out with 5 "with 3 left over, but you can't do anything with them".
10 R45	48 into 93 one time. Then 93 - 48 = 45. "48 can't go into 45, put 0 up." Then 48 x 0 = 0 and 45 - 0 = 45. (5 pupils)
1	Added 48 + 48 = 96 so decided there is one 48 in 93. Wrote 1 above 93 and left this as answer.
24 R200	8 x 3 = 24 (counted). Wrote 24 above 93 for quotient. Then 24 + 93 = 117. Placed this below 93. Then 83 + 117 = 200. Placed below 117. Then 117 + 200 = 417. Then subtract "7 - 0 = 0 ; 1 - 0 = 0 and 4 - 2 = 2".
1 R17	"48 go into 93 one time" ; 93 + 1 = 94 ; "48 go into 94 one time" ; subtracted (aside) 48 - 1 = 47 ; placed 47 under 94 and added for 141 ; added aside 96 + 96 for 172. And "that'll be too much, go 1 time". Placed 48 under 141 and subtracted for 17 (8 - 1 = 7 ; 4 - 4 = 0 ; bring down 1).
1 R3	Found 2 x 48 to be 96. Then 96 - 93 = 3 for remainder.
1 R44	Wrote 93 marks as she counted. Counted off 48 marks for quotient of 1. Miscounted remaining marks for R of 44.
1 R55	Decided 48 goes into 93 one time. Then 93 - 48 = 55. "3 from 8 = 5 and 4 from 9 = 5".
1 R55	48 goes into 93 one time ; then 93 - 48 = 55. "8 from 13 is 5 ; and 4 from 9 is 5".
13 1/48	"8 goes into 9 one time with 1 left over ; 4 goes into 13 three times with 1/48 left over."
1 R35	48 goes into 93 one time ; "8 out of 13 is 5 and 4 out of 7 is 3."
10	48 goes into 93 one time. 93 - 48 = 45. "How many 48's in 45? There's no 48's in 45."
1/48 45	By multiplication decided 48 goes into 93 one time. Then 93 - 48 = 45. Became confused, said "my answer is 1 and something, I don't know." Finally decided answer was "one 48, forty-five."
110	11 x 8 = 88 ; 93 - 88 = 5 ; "bring down the 4" from 48 to make 45, "and it won't go." Wrote 0 with 11 to make answer 110.
2 R03	Took 3 from 48 twice. Then 45 + 45 = 90. "Go back and put my 3 into it and make 93, would have a remainder of 3."
21 R5	"4 into 9 goes 2. Then 4 x 2 = 8 and 9 - 8 = 1". "8 into 13 goes 1. Then 8 x 1 = 8 and 13 - 8 = 5". (2 pupils)
1071 R6	48 into 93 goes 1 time ; 93 - 48 = 45 ; "48 into 45 goes 0 times, subtract and you still have 45 ; if you can't do that, start adding zeros." 48 goes into 450 seven times (after repeated multiplication and incorrect product of 404) ; then 450 - 404 = 54 and 48 into 54 one time with R6.

Divide 48/$\overline{93}$ (continued)

1.45	48 into 93 one time ; 93 - 48 = 45. "48 won't go into 45, I don't know how to do it." Decided answer was 1 and 45 tenths."
1723 R36	48 goes into 93 one time. 93 - 48 = 45. "Bring down 0." By repeated multiplication decided 48 goes into 450 seven times. Then 7 x 48 = 336 and 450 - 336 = 114. Bring down 0; by repeated multiplication of 48 [illegible] 7, 11, 14, 28, 20, 22, 23, decided 48 goes into 1140 twenty-three times. Then 1140 - 1104 = 36. (Note: was studying decimals at time of interview)
5 R910	48 into 93 goes 5 times ("48 is nearly 50, so 50, 60, 70, 80, 90, that's 5 times"). 5 x 8 = 40 ; 5 x 4 = 20 · wrote 2040 under 93 like this 93 / 2040 . "0 - 0 = 0 ; 4 into 3 = 1 ; 0 and 9 = 9 ; 2 x 0 = 0, so 0910.
21 R4	"4 goes into 9 twice" ; 2 x 4 = 8 and 93 - 8- = 13 : "8 into 13 goes 1 ; 1 x 8 = 8 ; 13 from 8 = 4."
21 R55	48 into 93 one time. 93 - 48 = 55 (8 from 13 = 5 ; 4 from 9 = 5).
2 R17	4 goes into 8 twice ; "2 x 8 = 16. Put down 6 and carry 1." 2 x 4 = 6, + 1 = 7. Then 93 - 75 = 17.
1	Wrote 93 marks, counted off 48 of them. "It will go 1 time. Wrote 93 under 93, and stopped.
1 R35	"48 into 93 one time ; 1 x 48 = 48 ; 13 take away 8 is 5, 9 take away 4 is 3."
1 23/24	"4 goes into 8 twice, about 1 less." 93 - 48 = 46 (8 from 13 is 6 ; 4 from 8 is 4).
No Answer	Decided 48 into 93 goes 5 times. Then 5 x 48 = 240. Decided then she shouldn't have multiplied 5 x 48, should have divided 5 into 48. Confused, stopped.
21 5/48	"4 goes into 9, two times ; 4 x 2 = 8 ; 9 - 8 = 1, bring down 3 ; 8 goes into 13 one time ; 8 x 1 = 8 ; 13 - 8 = 5."
2 R3	48 into 93 two times ; 2 x 48 = 96. Placed 96 below 93. Subtracted for 3.
5	Multiplied 48 by 5 aside, got 240. Placed 240 under 93 (4 under 3 ; and 2 under 9). Subtracted for 690 (bring down 0 ; 13 - 4 = 9 ; 13 marks on paper and crossed out 4 ; 9 - 2 = 6). Aside in another division 48 into 690 ; first 48 into 69 goes 21 times (counted on fingers from 48 to 69). 21 x 48 = 1008. Then 690- - 1008 = 5008 (bring down 8 ; 0 - 0 = 0 ; 9 - 0 = 0 ; 6 - 1 = 5). Stopped at this point.

Divide $74\overline{)6484}$

Answers

931	"7 goes into 64 nine times." 9 x 7 = 63 and 64 - 63 = 1. "Bring down 8 ; 7 goes into 18 two times, 2 x 7 = 14 and 18 - 14 = 4, bring down 4 ; 4 will go into 44 eleven times." Scratched out 2 of 92 in quotient and made final answer 931.
8071 R42	After repeated multiplication decided 74 goes into 648 eight times. Then 648 - 592 = 56. "74 won't go into 56 so you put up a 0 and bring down 4" ; 74 into 564 seven times ; 7 x 74 = 448 (7 x 4 = 28 and 7 x 7 = 42, + 2 = 44). 564 - 448 = 116 ; 74 into 116 one time remainder 42.
574	Subtracted 74 from 648 and got 574. "If that is the right number, I would put 5 over 8 because that is the last number used, and 7 would go over the 4, but I don't know what to do with the 4" (of 574). Stopped.
717 46/74	74 into 648 ; seven times ; 7 x 74 = 518 ; 648 - 518 = 130 ; 74 into 130 one time and 130 - 74 = 56 ; bring down 4 ; 74 into 564, seven times and 564 - 518 = 46 for remainder of 46/74.
8 R12404	74 goes into 648 eight times (repeated multiplication) ; 8 x 74 = 592. Wrote 5920 under 6484 and added for remainder of 12404.
8 R36	74 into 648, eight times ; 8 x 74 = 592 and 648 - 592 = 36 ("8 take away 2 is 6 and 9 from 14 is 3").
8 R46	74 into 648 eight times ; 8 x 74 = 592 ; 648 - 592 = 46 ("2 from 8 is 6 ; 9 from 14 is 5, so I borrow one and that makes that a 4").
43 R46	Added and subtracted 74's until reached 592. Decided had used four 74's. Wrote 4 in quotient and 592 under 648. Subtracted for 56, brought down 4. Subtracted 74 from 592, got 518. So this meant 74 goes into 564, three times (one less than 4). Finally 564 - 518 = 46 for remainder.
81 R490	74 into 648 eight times (used repeated addition). 648 - 592 = 56 brought down 4. "74 goes into 84 one time" wrote 74 under 564 and 564 - 74 = 490.
81	74 into 648 goes 8 (repeated multiplication). "74 goes into 84 one time." Stopped for an answer of 81.
126 R46	74 x 3 = 222 ; 222 x 3 = 666 ; 666 x 3 = 5994. Had multiplied 74 by 3 four times, so 74 goes into 6484 twelve times. 6484 - 5994 = 496. Then 74 + 74 = 148 ; 148 + 148 = 296 ; 296 + 148 = 444. So 74 goes into 496 six times with remainder 46.
8 R374	74 + 74 = 148 ; 148 + 148 = 296 ; 296 + 296 = 592 ; 592 + 592 = 1184 ; 1184 + 1184 = 2368 ; 2368 + 2368 = 4936 ; 4936 + 4936 = 9872 ; "that's too large, so I'll use 4936." Counted multiplications (1 for 74's ; 2 for 148's ; 2 for 296's ; 2 for 592's and 1 for 1184's).

Divide $74/\overline{6484}$ (continued)

8 R374 (cont.)	Wrote 8 as quotient; placed 4936 under 6484 and added for 11420. Multiplied 74 by 8 for 98 (7 x 4 = 28, 7 and 2 is 9). Wrote 98 under 11420, subtracted for 11322 ; "74 into that, ten times" and 11322 - 1184- = 522 ; "74 goes into that only 3 times". Wrote 296 under 522. Subtracted for remainder of 374.
6 R64	"74 goes into 648 seven times because 7 x 4 is 28". Then 7 x 74 = 658 (thought 28 + 28 = 56, and 7 x 7 = 56, + 7 = 63). "That's ten too high, so try 74 x 6" got 584. Then 584 + 55 ; then 584 + 64 which gave 648. "So 74 goes into 6484 six times." Put 584 under 648, subtracted for 64. "74 can't go into 64 so the answer is 6 with a remainder of 64."
6 R564	After repeated multiplications and subtractions decided 74 goes into 648 six times. Wrote 6 as quotient and 592 under 648. Subtracted for 56 ; brought down 4. Tried several multiplications, subtractions and additions for 564 ÷ 74. Gave up.
5334 R46	Multiplied 74 by 2, 3, and 5. 74 goes into 648, five times ; 648 - 370 = 278. Brought down 4 ; 74 into 278, three times, 2784 - 222- = 564 ; wrote another 3 in quotient. Then 564 - 222 = 342. Wrote 4 in quotient. Then 342 - 296 = 46 remainder.
8 R56	Repeated multiplications and additions. 74 goes into 648, eight times ; 648 - 592 = 56.
81 R49	"74 cannot go into 4 ; 74 go into 84 once" ; 84 - 74 = 10, bring down 64 for 6410 ; mark out 0 ; 74 into 641 eight times ; 641 - 592 = 49.
8 R56	74 into 648, eight times ; 648 - 592 = 56, brought down 4 ; then 74 into 564, seven times. Did not write 7 in quotient. Finally 564 - 518 = 56.
5261 R102	Multiplied 74 by 12, by 24, by 52, by 61. "Put 52 over there (in quotient) because 3848 (74 x 52) is the closest to 6484." Then 6484 - 3848 = 2636. Placed 2734 (incorrect product of 74 x 61) under 2636. Then 2636 - 2734 = 102.
1 R10	"74 can't go into 64. I guess I'll see how many times it'll go into 84, one time I think, and 75, 76, 77, 78 ... 84 (made 10 marks as counted), so answer is 1 remainder 10."
81 R49	"74 cannot go into 1 ; 74 go into 84 one time" 84 - 74 = 10. "74 cannot go into 1 ; 74 cannot go into 41, so I bring down the whole 64." 74 into 641 eight times. 641 - 592 = 49.
No answer	74 x 3 = 282 ; 74 x 2 = 148 ; 148 + 282 = 430 ; 74 x 12 = 888 ; 74 x 17 = 1328 ; 74 x 23 = 1702 ; 74 x 38 = 3482 ; 74 x 41 = 3034. Gave up.

Divide 74/6484 (continued)

89 R102 Multiplied by 4, 6, 7, 8, and 9. Decided 74 into 648 eight times ; 648-592 = 76 ; ("8 can't go into 2 so I borrow 1, then 18 - 2 = 16 ; 9 - 3 = 6, plus 1 is 7) 74 into 764 nine times. 764 - 666 = 102. (4 minus 6 = 2, 6 - 6 = 0 ; 7 - 6 = 1)

800 R4 74 into 648, eight times (by estimate). 8 x 74 = 648. (Without multiplying) 648 - 648 = 000. bring down 4 ; 74 into 4 zero times, put 0 in quotient ; 0004 - 0 = 04 ; 74 goes into 4 zero times - another zero in quotient ; 04 - 4 = 4 so R 4.

87 R26 Error in subtracting 643 - 592. Got 544 instead of 564. Then 544 - 518 = 26 instead of 46.

80 R0 74 into 648, eight times. 649 - 592 = 156 ("2 from 8 is 6 ; 9 from 5 is 4 ; 6 - 5 = 1"). Crossed out 156. Wrote 56 below. "74 into 56 is 0, put 0 up top" ; wrote 74 under 56 ; "can't subtract 74 from 56 so the answer is 0."

1511R924 Made these multiplications and additions. 74 x 6 = 444, added three 74's for 666 ; 666 x 4 = 2664 ; 2664 x 3 = 7992 ; 2264 x 2 = 5228. Wrote 15 in quotient. Then 6484 - 5228 = 1256. Multiplied 666 x 2 for 1332. Wrote 11 in quotient, then 1256 - 1332 = 924.

8168 R12 7 goes into 64 eight times ; 64 - 56 = 12 ; 7 goes into 12 one time ; 12 - 7 = 4 ; 7 won't go into 4 so bring down 8 ; 7 go into 48 six times. 48 - 42 = 6, bring down 4 · 7 goes into 64 eight times ; 64 - 56 = 12.

88 R0 "74 won't go into 64, so 74 into 648." By multiplication tried 5, 7 and 9. Decided on 8. Wrote 592 under 592 (below 6484) and subtracted for 0. Wrote quotient as 88.

710 R6968 "74 into 64 is 10". Wrote 484 under 484 of 6484. "74 into 48 is 7" (no reason). Added 6484 + 484 = 6968.

82 R4 "7 goes into 64 eight times" wrote 8 above 4 of 64. "4 goes into 8 two times' wrote 2 above 8 of 648. Wrote 648 under 648, "bring down 4".

760 By repeated addition of 74, to 592, decided 74 goes into 648 seven times. Then 648 - 592 = 56. Bring down 4 ; 74 into 564 six times because 518 is between 592 and 444. Then 564 - 518 = 46 ; "74 won't go into 46, therefore 0, and 0 x 74 = 0. No remainder.

87 R66 Error in remainder. 564 - 498 (7 x 74 = "7 x 4 is 8 and 7 x 7 is 49").

86 R120 74 into 648, eight times and 648 - 592 = 56. 74 into 564, six times ; 6 x 74 = 444 and 564 - 444 = 120.

761 R10 74 into 648 seven times, then 648 - 596 = 52 bring down 4 ; 74 into 524, six times. 524 - 444 = 80 ; 74 into 80 goes once ; 80 - 74 = 10 ("4 from 0 is 0, 8 from 7 is 1").

852 R46 74 into 648, eight times ; 648 - 592 = 56. 74 into 564, five times ; 564 - 370 = 194 ; 74 into 194, two times ; 194 - 148 = 46.

Divide 74/6484 (continued)

870 R46 Error in final step. 564 - 518 = 46. Then 74 into 46. Zero times placed 0 in quotient and under 46.

852 R48 74 into 648, eight times ; 648 - 592 = 56. Bring down 4. 74 into 564, five times ; 564 - 370 = 194 ; 74 into 194, two times ; 194 - 148 = 48.

8 74 into 648, eight times, 648 - 592 = 146. Tried 74 x 3 and 74 x 2, stopped.

90 R24 74 into 648, nine times ; 9 x 74 = 646, and 648 - 646 = 2, bring down 4. 74 won't go into 24 ; so put 0 in quotient and 24 - 0 = 24.

80 R56 74 into 648, eight times ; 648 - 592 = 56. 74 won't go into 56 so put 0 in quotient ; 56 - 0 = 56.

71 R56 74 into 648, seven times ; 648 - 518 = 130. 74 into 130 one time ; 130 - 74 = 56.

91 "7 won't go into 6. So 7 into 64 goes 9." (7 x 7 = 49, +7 = 56, 57, 58 ... 63). "Put 9 over 6, add 1 to 4 (of 74), make it 5, 5 into 8 goes 1, have 3 left over." Stopped, did not know what to do with 3. "Could put it over 4 (of 84) but that would make 7 and there are no numbers left to divide by."

9776 7 into 64 nine times, 9 x 74 = 666. Placed 666 under 464 of 6484. Subtracted 6484 - 666 = 5818 ; 7 into 58 goes 7 ; 7 x 74 = 518. Wrote 518 under 818 of 5818. Subtracted for 5300 ; 7 into 53, seven ; 7 x 74 = 518 ; 5300 - 518 = 4782 ; 7 into 47 goes 6.

81 R24 74 into 648, eight times ; 8 x 74 = 572 (8 x 4 = 32, 8 x 7 = 54, + 3 = 57). 648 - 572 = 76 ; 74 into 76 once, 76 - 74 = 2, bring down 4.

87 R45 74 into 648, eight times ; 8 x 74 = 582 (8 x 4 = 32 ; 8 x 7 = 56, + 2 = 58). 648 - 582 = 56 ("2 from 8 = 6, 8 - 14 = 5") bring down 4 ; 74 into 564, seven times, 7 x 74 = 518 ; 564 - 518 = 45.

86 74 into 648, eight times ; 8 x 74 = 602 (4 x 8 = 32, 7 x 8 = 57, + 3 = 60). 648 - 602 = 46, bring down 4 ; 74 into 464, six times. Wrote 464 under 464, drew line and stopped.

871 R22 74 into 648, eight times, 8 x 74 = 592 ; 648 - 592 = 56. Bring down 4 ; 74 into 564, seven times, 7 x 74 = 468 (7 x 4 = 28, carry 2, 7 x 7 = 44, + 2 = 46). 564 - 468 = 96. 74 into 96, once ; 96 - 74 = 22.

970 R2 74 into 648, nine times ; 9 x 74 = 592 ; 648 - 592 = 56, bring down 4 ; 74 into 564 seven times ; 7 x 74 = 518 ; 564 - 518 = 46. "74 won't go in 46" so put 0 in quotient. Then "0 x 74 = 74," wrote 74 under 46. Said "6 - 4 = 2."

91 R30 74 into 648 nine times ; 9 x 74 = 638 (9 x 4 = 28 ; 9 x 7 = 63). 648 - 638 = 10, bring down 4 ; 74 into 104 once, 104 - 74 = 30.

9 R818 74 into 6484 nine times, 9 x 74 = 666. Then 6484 - 666 = 818.

Divide 74/6484 (cont...

No answer — Rounded 74 to 70 and 6484 into 6080. "70 won't go into 60 but will go into 608"; then 70 x 90, by 80, by 70 and by 100. Decided wouldn't be a whole number so 74 x 95. Decided it must be less than 90 because 6300 is more than 6080. Then 90 x 65. Stopped.

Divide 15/7590 (Two schools)

<u>Answers</u>

560 — Placed 5 above 5 of 75 and 6 above 9. Subtracted 75 - 75 and 90 - 90 = 0. Affixed zero to 56 because "there is nothing else to multiply by."

560 — After 75 ÷ 15 = 5 and 90 ÷ 15 = 6 for 56, affixed a 0 because "ain't nothing more to multiply by to get no more numbers."

56 — Placed 5 above 5 and 6 above 0 of 7590. After 75 - 75 said "bring down your 9 and 15 won't go into 9 so bring down 0 ; 6 x 15 = 90, so the answer is 56." (8 pupils)

5459 5/15 — "15 into 75 five times." Placed in quotient and by 9 in dividend ; "15 goes into 59 three times because 3 x 15 = 45." Put 45 in quotient making 545 ; 59 - 45 = 14. Placed it beside 0 of 7590. "15 can't go into 14, so 15 goes into 140 nine times with 5/15 left over.

560 — 15 into 75 five times ; 75 - 75 = 0. "15 won't go into 0 so bring down 9 ; 15 won't go into 9 so bring down 90 15 into 90 six times ; 90 - 90 = 0 ; 15 goes into 0 zero times so 560. (2 pupils)

56 — 15 into 75 five times 15 into 90, six times. Wrote 5 above 5 and 6 above 0 of 7590. Puzzled by vacant space between 5 and 6 but left answer at 56.

56 — 15 into 75 = 5. Subtracted 7590 - 7500 = 90. "There are six 15's in 90."

5 — Wanted to write 5 above 5 and 6 above 9 of 7590. Decided this was wrong. Tried 15 into 59 and 15 into 759. Left 5 as answer. Confused with short division.

56 — Wrote 5 above 5 of 7590. Then 15 into 90 = 6. Moved 5 above 9 and placed 6 above 0, "because you have only two numbers" (in quotient).

7118 — 1 into 7, seven times ; 7 x 1 = 7 and 75 - 7- = 05. 5 goes into 5 once ; 05 - 5 = 0. Bring down 9 ; 5 into 9 once ; 09 - 5 = 4. Bring down 0 ; 5 into 40 eight.

4 R20 — "15 into 75 four times. How many times can 4 go into 15? 4 into 15 = 3." Put 3 under 5 of 7590. Brought down 9, wrote by 3 to make 39 under 59 of 7590. Subtracted 59 - 39 = 20.

56 R0 — 15 into 75 = 5 ; 15 into 90 = 6. 90 - 90 = 0.

APPENDIX B

Wrong Answers--Fractions

3/4 + 5/2 = ___

Answers

8/4 = 2	"5 + 3 = 8. You don't add the bottom numbers because 2 will go into 4." (23 pupils)
8/6	3 and 5 is 8 ; 4 and 2 is 6. (May or may not be reduced.) (36 pupils)
8/8	5 + 3 = 8 ; 2 plus 4 is 8. (3 pupils)
68	3 and 5 is 8 ; 4 and 2 is 6.
15/8	5 and 3 is 15 ; 4 and 2 is 8. (2 pupils)
7/7	3 + 4 = 7 ; 2 + 5 = 7. (2 pupils)
7/6	3 + 5 is 7 ; 4 and 2 is 6.
3 1/8	Chose 8 as C.D. Then 4 x 3 = 12, so 3/4 = 12/8 ; 5 x 2 is 10, so 5/2 = 10/8 ; 12/8 + 10/8 = 22/8 = 3 1/8 (3 x 7 = 21, with 1 left over).
86	5 + 3 = 8 ; 4 + 2 = 6. (2 pupils)
2/8	Chose 8 as C.D. "8 goes into 2 four times and 4 goes into 5 one time." So 5/2 became 1/8. "8 goes into 4, two times, 2 will go into 3 one time," so 3/4 became 1/8. Then 1/8 + 1/8 = 2/8.
16/8 = 2	Chose 8 as C.D. Then 3/4 = 6/8 and 5/2 = 10/8, and 6/8 + 10/8 = 16/8.
14	"5 over 2 is 7 ; 3 over 4 is 7 ; 7 + 7 = 14."
2	3/4 = 3/4 and 5/2 = 5/4. Then 3 + 5 = 8 over 4 for 8/4 = 2.
2 1/4	Got 13/4 correctly. Decided, by counting, there are two 4's in 13 with one left over.
3/2	5/2 = 5/4 ; 3/4 = 3/4. Then 5 + 3 = 8 over 4 and 8/4 = 3/2.
3	3/4 = 3/4 ; 5/2 = 2 1/2 ; chose 8 as C.D. For 3/4 said "4 goes into 8 two times and 3 goes into 8 two times." So wrote 2/2 for 3/4. Then "2 goes into 8 four times, and 1 goes into 8 eight times" so wrote 8/4 for 2 1/2. Then divided $\overset{2}{4\overline{)8}}$. Added this 2 to 1 from 2/2 of 3/4 for answer of 3.
3 3/2	Wrote 5/2 as 20/8 and 3/4 as 6/8, added for 26/8. 26 ÷ 8 = 3 R2 wrote as 3 3/2.
24/4	Chose 4 as C.D. Then for 3/4 said "4 times 1 equals 4, and 1 plus 3 is 4" so 4/4. For 5/2 said "2 x 2 = 4 and 4 x 5 = 20" so 20/4. Then 4/4 + 20/4 = 24/4.
4 1/4	Got sum of 13/4 correctly. Then "4 into 13 goes 4 with 1 left over."
59	"4 and 5 is 9 ; 3 and 2 is 5. Answer would be 95" but wrote as 59.

3/4 + 5/2 = ___ (continued)

3 1/2	Wrote 3/4 as 3/4 and 5/2 as 2 1/2 : then 1/2 as 3/4 ("2 goes into 4 two times plus 1 is 3"). Then 3/4 + 3/4 = 6/4 = 1 1/2 and 2 + 1 1/2 = 3 1/2.
8/2	Chose 2 as C.D. because "2 will go into 4 and 2, equal times." Then 5 + 3 = 8 for 8/2.

3/8 + 7/8 = ___

Answers

11/16	7 + 3 = 11 and 8 + 8 = 16. Got 7 + 3 = 11 because she remembered 7 + 4 = 12.
10/16	3 + 7 = 10 and 8 + 8 = 16. (37 pupils)
2/2	Got 10/8 correctly. Wrote as 2/2 because "2 will go into 10 and 2 will go into 8". (2 pupils)
1 3/8	[illegible] + [illegible] = 11/8 = 1 3/8. (2 pupils)
11/15	8 and 3 is 11 (counted), 7 and 8 is 15 (counted). (2 pupils)
2 1/8	Got 10/8 correctly. "8 goes into 10 one time," wrote 1/8. "Remainder is 2" so 2 1/8.
26	"7 over 8 is 15 ; 3 over 8 is 11 ; 15 + 11 = 26."
5 1/8	Got 10/8 correctly, then "the numerator is bigger than the denominator so you got to break it down, that'll be 1 2/8 = 5 1/8 because 2 goes into 8 four times and 4 + 1 = 5." Could not explain the 1/8.
21/8	3 x 7 = 21 over 8.

5 7/8 + 2 1/2 = ___

Answers

8	5 + 2 = 7 and 7 + 1 = 8 and 8 is C.D. because 2 and 8 will go into 8.
7 8/10	5 + 2 = 7 ; 7 + 1 = 8 ; 8 + 2 = 10. (23 pupils)
Incomplete	7 + 1 = 8 ; 8 + 2 = 10. "That's not right." Could not proceed further.
8 3/4	5 + 2 = 7, thought 1/2 = 4/8. Then "7 + 4 = 11." "That would be 1 3/4 ; so the answer is 8 3/4."
52/10	5 7/8 = 47/8 ; 2 1/2 = 5/2 ; wrote 47/8 + 5/2 = 52/10, 47 + 5 = 52 and 8 + 2 = 10. (7 pupils)
3 1/16	Wrote vertically, chose 16 as C.D. Then 7/8 = 56/16 because 8 x 7 = 56 and 1/2 = 2/16 because 2 x 1 = 2. Then 56/16 + 2/16 = 58/16 ; "16 in 58 goes 3 times. Remainder will be 1." So 3 1/16.

5 7/8 + 2 1/2 = ____ (continued)

20/5	8 + 5 = 13 (counted) ; 13 + 7 = 20 (counted). Wrote 20 for numerator ; 2 + 2 = 4 and 1 is 5. Wrote 5 in denominator.
7 8/8	Wrote 5 7/8 for 5 7/8 and 2 1/8 for 2 1/2. Then 5 + 2 = 7 ; and 7/8 + 1/8 = 8/8.
8 5/8	Wrote vertically ; then 5 14/16 for 5 7/8 and 2 8/16 for 2 1/2 ; then 5 + 2 = 7 and 14 + 8 = 26 ; 7 26/16 = 8 10/16 = 8 5/8.
7 8/5	Tape became fouled. Explanation was not recorded.
7 13/8	Wrote vertically 5 7/8 for 5 7/8 and 2 6/8 for 2 1/2. Added for 7 13/8.
1 6/16	Wrote 14/16 for 7/8 and 8/16 for 1/2. Then 14/16 + 8/16 = 22/16 = 1 6/16. Ignored whole numbers.
7 7/16	Wrote 14/16 for 7/8 and 8/16 for 1/2 ; subtracted (counted fingers) and got 7, so 7/16. Then 5 + 2 = 7.
7 13/8	Chose 8 as C.D. Wrote 8/8 for 7/8. ("8 into 8 one time and 1 + 7 = 8") ; 5/8 for 1/2 ("2 into 8 four times, and 4 + 1 = 5") ; 8/8 + 5/8 = 13/8 and 5 + 2 = 7.
8	Chose 8 as C.D. Wrote 7/8 for 7/8 ; and 1/8 for 1/2, then 5 + 2 = 7 and 7/8 + 1/8 = 8/8 ; 7 8/8 = 8.
7 8/8	"5 + 2 = 7 ; 7 + 1 = 8 and the determiner is 8" so 7 8/8. (3 pupils)
52/8	Wrote 5 7/8 as 47/8 and 2 1/2 as 5/2. Then 47 + 5 = 52 and "common denominator for this would be 8." Wrote 52/8 as answer.
6 8/10	"5 + 2 = 6 ; 7 + 1 = 8 and 8 + 2 = 10."
2 7/20	Wrote 47/8 for 5 7/8 and 20/8 for 2 1/2 ; then sum = 47/20 = 2 7/20.
4	5 7/8 = 20 (8 + 5 = 13 and 13 + 7 = 20) ; then 2 1/2 = 5 (2 + 2 + 1 = 5) ; so 20/5 = 4.
8 4/8	Wrote 7/8 for 7/8 and 4/8 for 1/2 ; then 7 + 4 = 11 and 11/8 = 1 4/8 (11 - 8 = 4).
7 16/16	Wrote 5 7/8 as 5 14/16 and 2 1/2 as 2 2/16 ; (multiplied 7 of 7/8 by 2 so multiplied 1 of 1/2 by 2) ; 5 14/16 + 2 2/16 = 7 16/16.
25	"7 + 8 = 15 and 5 is 20 : 1 over 2 is 3 and 2 more is 5", 20 + 5 = 25.
8 5/8	Wrote 5 8/8 for 5 7/8 (8 into 8 one time ; 1 + 7 = 8). Wrote 2 5/8 for 2 1/2 (2 into 8 four times ; 4 + 1 = 5). 5 8/8 = 6 and 2 + 6 = 8, "bring down 5/8".
7	Wrote 5 7/8 as 5 7/4 ("because 2 will go into 8 four times"). Wrote 2 1/2 as 2 1/4 ("because 2 goes into 4 two times"). Added 5 + 2 = 7 and stopped. Didn't know what to do with 7/4 + 1/4.
52	"8 times 5 is 40, + 7 = 47 ; 2 times 2 = 4, + 1 = 5 ; 47 + 5 = 52.

2/3 + 1/2 = ____

Answers

3/6	2 + 1 = 3 ; 2 and 3 will go into 6 so 6 is C.D. (2 pupils)
3/5	2 + 1 = 3 and 3 + 2 = 5. (31 pupils)
5/3	2 + 3 = 5 and 2 + 1 = 3. (2 pupils)
1 2/6	Chose 6 for C.D. Wrote 6/6 for 2/3 (2 x 3 = 6) and 2/6 for 1/2 (2 x 1 = 2) ; 6/6 + 2/6 = 8/6 = 1 2/6.
1/2	Wrote 2/6 for 2/3 and 1/6 for 1/2. Then 2/6 + 1/6 = 3/6 = 1/2.
8/6	"2 (of 1/2) will go in 6 three times ; 3 + 1 = 4." Wrote 4/6 for 2/3 ; "3 (of 2/3) will go in 6 two times ; 2 + 2 = 4." Wrote 4/6 for 1/2. Then 4/6 + 4/6 = 8/6.
3/1	"Need a number that will go into 3 and 1." Chose 1 as denominator. Then 2 + 1 = 3 for numerator.
1/2	"2 + 1 = 3 and 3 + 2 = 6" ; then 3/6 = 1/2.
1 2/6	Wrote 4/6 for 2/3 (3 goes into 6 two times and 2 + 2 = 4). Wrote 4/6 for 1/2 (2 goes into 6 three times and 1 + 3 = 4). 4/6 + 4/6 = 8/6 = 1 2/6.
3/6	Chose 6 as C.D. Wrote 2/6 for 2/3 and 1/6 for 1/2. Then 2/6 + 1/6 = 3/6.
3/6	"3 can't go into 2 and 2 can't go into 3, so I can multiply them together" (gave 6 for denominator). Then 2 + 1 = 3.
3/6	2 + 1 = 2 and 3 x 2 = 6. (3 pupils)
8/6	Wrote 2/3 as 4/6 (three goes into 6 two times, 2 x 2 = 4). Wrote 1/2 as 4/6 (2 goes into 6 four times and 4 x 1 = 4), then 4/6 + 4/6 = 8/6.
Incomplete	2+ 1 = 3 for numerator. Could not remember how to find C.D.
1/2	Wrote as 2/3 + 1/2. Cancelled 2 into 2 for 1 and 1. Then 1 + 1 = 2 and 3 + 1 = 4 for 2/4 = 1/2.

3/4 - 1/2 = ____

Answers

2/2 or 1	3 - 1 = 2 and 4 - 2 = 2. (44 pupils)
2/2	Chose 2 as C.D. "Need to find number that will go into 4 because the bottom number has to be the same as this (2 of 1/2)." Then 3 - 1 = 2.
Incomplete	Wrote 2/4 for 3/4 ("4 goes into 4 one time and 3 - 1 = 2") and 2/4 for 1/2 ("2 goes into 4 two times and 2 x 1 = 2"). Couldn't go further.

3/4 - 1/2 = ___ (continued)

Incomplete	Rewrote vertically, "you have to make 4 and 2 even ; 2 won't go into 3 evenly, 8 won't go, 9 won't go into 4, try 12 ; 3 divided by 12 goes 4 times, 4 divided by 12 goes 3 times ; 1 divided by 12, it will go 1 time, 1 goes into 2 1 time and 1 left over. It won't go evenly because there is 1 left over. You have to try another number." Stopped.
0/4	Chose 4 as C.D. 4 into 4 one time, 1 x 3 = 3. So 3/4 = 3/4 ; 2 goes into 4 three times 3 x 1 = 3 so 1/2 = 3/4 ; then 3 - 3 = 0.
5/4	1/2 = 2/4 and 3/4 = 3/4, then 3 + 2 = 5. "When you subtract, you don't subtract, you add the opposite."
9/8	"Have to make the 3 a 13 ; have to make 1 a 10." Then 13 - 4 = 9 and 10 - 2 = 8.
1/1	"3 subtract 4 is 1" (numerator) ; "1 subtract 2 leaves 1" (denominator).
0/0	"3 won't go into 1" wrote 0 for numerator. "4 won't go into 2" wrote 0 for denominator.
1 2/8	Chose 8 as C.D. Wrote 3/4 as 12/8 (3 x 4 = 12) and 1/2 as 2/8 (1 x 2 = 2). Then 12/8 - 2/8 = 10/8 = 1 2/8.
0/8	Chose 8 as C.D. Wrote 3/4 as 1/8 ("8 will go into 4 two times and 2 will go into 3 one time"). Wrote 1/2 as 1/8 ("8 will go into 2 four times, 4 will go into 1 one time"), 1/8 - 1/8 = 0/8.
1/2	Wrote 3/4 as 3/4 and 1/2 as 1/4. Then 3/4 - 1/4 = 2/4 = 1/2.
Incomplete	First chose 8 as C.D. wrote 1/8 for 3/4 ("4 goes into 8 two times and 3 from 2 is 1"). Wrote 3/8 for 1/2 ("8 goes into 2 four times and 4 take away 1 is 3") ; "that ain't gonna work because you can't take 3 from 1". Tried C.D. of 16. By same process got 1/16 for 3/4 and 7/16 for 1/2. Still could not subtract 7 from 1 so gave up.
2/4	Chose 4 as C.D. Wrote 3/4 for 3/4 and 1/4 for 1/2. Then 3/4 - 1/4 = 2/4.
2/4	Chose 4 as C.D. Then 3 - 1 = 2.
0/4	Chose 4 as C.D. Wrote 4/4 for 3/4 ("4 goes into 4 one time, 1 plus 3 is 4"). Wrote 5/4 for 1/2 ("2 goes into 4 two times, plus the 1 is 5"). "Can't take 5 from 4, so I borrow one from the denominator make it 5/4". Then 5/4 - 5/4 = 0/4.
22	"2 take away 4 is 2, 1 take away 3 is 2."
2/6	Chose 6 for denominator, 3 - 1 = 2 for numerator. Then "2 will go into 6 four times, and 4 will go into 6 with 2 left over."
4/8	Wrote 6/8 for 3/4 ("4 x 2 = 8 so 3 x 2 = 6"). Wrote 2/8 for 1/2 (1 x 2 = 2 because he multiplied the 3 of 3/4 by 2 he must use same number here). Then 6/8 - 2/8 = 4/8.

3/4 - 1/2 = ___ (continued)

11	"3 over 4 leaves 1 ; 1 from 1/2 leaves 1."
1/2	Chose 4 as C.D. Wrote $\frac{}{4}$ and $\frac{}{4}$. "Put my 3 here, minus 1/4, that would be 2/4 or 1/2."
1/4	Wrote 4/4 for 3/4 ("4 goes into 4 one time, 3 + 1 = 4"). Wrote 3/4 for 1/2 ("2 will go into 4 two times, 2 + 1 = 3") ; 4 - 3 = 1 "bring down the 4."

Subtract 8 2/5
<u>4 3/10</u> .

<u>Answers</u>

21/10	Wrote 8 2/5 as 42/5, then as 84/10. Wrote 4 3/10 as 43/10. Then 84 - 43 = 21 ("4 minus 3 = 1 and 8 - 4 gives 2").
4 1/5	8 - 4 = 4 ; 3 - 2 = 1 and 5 from 10 is 5. (12 pupils)
4 7/2	"4 from 8 is 4 ; can't subtract 5 (of 2/5) from 2 so have to make the 2 a 12 and 12 - 5 = 7" ; "can't get 10 from 3 so make the 3 a 12." (Thought she had borrowed 1 from 3 of 3/10 to make the 2 of 2/5 into 12 leaving 2/10 which she now made 12/10). Now had 8 12/5 - 4 12/10 arranged vertically, then 12 - 5 = 7 and 12 - 10 = 2.
4 7/3	"4 subtract 8 leaves 4 ; 3 subtract 10 leaves 7 (counted fingers) ; 2 subtract 5 leaves 3."
4 1/2	8 - 4 = 4 ; "2 goes into 3 one time, 5 goes into 10 two times."
5	Chose 20 as C.D. Wrote 10/20 for 2/5 (wrote 20 as denominator and 5 x 2 = 10). Wrote 30/20 for 3/10 (wrote 30 as denominator and 3 x 10 = 30). Then 4 from 8 is 4 and 10 from 30 is 20 so 4 20/20 = 5.
Incomplete	"8 take away 4 is 4", "2 take away ???" stopped.
394	"2 over 5 would leave 3 ; 3 over 10 would leave 9 ; 4 from 8 would leave 4."
Incomplete	Chose 10 as C.D. Wrote $\frac{}{10}$ and $\frac{}{10}$. "Bring over the 2 and the 3" for 2/10 and 3/10 ; "4 from 8 leaves 4 ; "you can't subtract 3 from 2, you have to multiply it someway, I can't remember what it is."
2 1/0	"3 from 2 you cannot take, go over to 8 and regroup a whole 2. That leaves 4 (in place of 2 of 2/5) and that leaves 7 (in place of 8)." Then "3 from 4 leaves 1; "10 from 5 you cannot take so you go to 7 and regroup a 5, that leaves 10 (in place of 5 of 2/5), and that leaves 6" (in place of 7 replacing original 8). Then 6 4/10 - 4 3/10 = 2 1/0 (6 - 4 leaves 2, 4 - 3 leaves 1, and 10 - 10 leaves 0).

Subtract 8 2/5 over 4 3/10 (continued)

4 0/10	Chose 10 as C.D. Wrote 8 4/10 for 8 2/5 ("5 goes into 10 two times, 2 + 2 = 4"); wrote 4 4/10 for 4 3/10 ("10 will go into 10 one time and 1 + 3 = 4"). Then 8 4/10 - 4 4/10 = 4 0/10. ("8 - 4 = 4, 4 - 4 = 0, bring down 10").
3 9/10	Wrote 8 2/5 as 8 2/10 and 4 3/10 as 4 3/10. Marked out 8 made it 7 ; made 2/10 into 12/10. Then 7 - 4 = 3 and 12/10 - 3/10 = 9/10.
3 9/5	10 will not go into 5 so change 8 to a 7 and 2 (of 2/5) to 12. "Then I borrow one from 12 making it 11, and making 5 (of 2/5) a 15" ; so 7 11/15 - 4 3/10 = 3 9/5 ("4 from 7 = 3 ; 10 from 15 = 5 ; 3 from 11 = 9").
76/40	Wrote 80/10 for 8 and 4/10 for 2/5. Subtracted 4 from 80 for 76/10. Wrote 4 as 80/20 and 3/10 as 6/20; subtracted 6 from 8 and got 14 for 14/20. Then 76/10 over 14/20. Converted to 304/40 over 28/40. Subtracted 304 - 28 = 076. So answer is 76/40.
4 0/10	Wrote 4/10 for 2/5 ("5 into 10 twice, 2 + 2 = 4"). Wrote 4/10 for 3/10 (10 into 10 once 1 + 3 = 4). Then 8 - 4 = 4 and 4/10 - 4/10 = 0/10.
9/10	Borrowed 1 from 8, made it a 7. Made 2 of 2/5 a 12. Then 12/5 - 3/10 = 9/10.
3 9/5	Borrowed 1 from 8, made it 7 and 2 of 2/5 into 12. Then 7 12/5 - 4 3/10 = 3 9/5 ("4 take away 7 is 3 ; 3 take away 12 is 9, and 10 take away 5 is 5").
3 9/5	"8 take away 4 is 4, you can't take 2 from 3 so you borrow from 4 (the difference number)". Then "12 take away 3 is 9 and 5 take away 10 is 5."
2 9/5	Cross out 8 make it a 7 and 2 of 2/5 a 12. Cross out 7 make it a 6 and 5 of 2/5 a 15. Then 6 12/15 - 4 3/10 = 2 9/5. (6 - 4 = 2 ; 12 - 3 = 9 ; 15 - 10 = 5).
4 9/5	"8 take away 4 is 4." "Can't take 3 from 2, borrow from 8 make it a 7, and 2 (of 2/5) a 3, then 3 take away 3 is 0 ; 5 can't take away 10 ; borrow again, that's a 6" (crossed out 7), made 5 (of 2/5) a 15. Then 12/15 - 3/10 = 9/5 (had previously written 0/5 and now corrected to 9/5).
2 9/5	3 from 2 won't go so cancel 8 and make it a 7. Make the 2 (of 2/5) a 12 ; 10 from 5 won't go so cancel the 7 make it a 6 and this 5 (of 2/5) a 15. "Then 6 12/15 - 4 3/10 = 2 9/5 (4 from 6 is 2, 3 from 12 is 9 ; 10 goes into 15 once with 5 left)."

Subtract 8 2/5
4 3/10 (continued)

Incomplete	"5 from 10 is 5, 4 from 8 is 4, and 2 from . . ." "Oh! can't take 8 from 4 or 10 from 5." Then "5 from 10 is 5, 2 from 3 is 1, have to find a way to make the 4 bigger because you can't take 8 from 4." Stopped.
3 11/5	"4 from 8 is 4, 2 into 3 wouldn't go, you need a proper fraction, borrow from next number over, borrow from here (the 8) ; that'd be 7" , then put 1 from 8 with 2 of 2/5 to make 12/5 ; then 7 12/5 - 4 3/10 = 3 11/5.
4 6/50	Wrote 2/5 as 8/50. ("5 will go into 50 ten times and 10 from 2 is 8"). Wrote 3/10 as 2/50 ("10 will go into 50 five times and 5 from 3 is 2"). Then 8 - 4 = 4 and 8 - 2 = 6 for 4 6/50.
4 2/10	Chose 10 as C.D. Wrote 0/10 for 2/5 (5 goes into 10 two times and 2 take away 2 is 0"). Wrote 2/10 for 3/10 ("10 goes into 10 one time, and 1 take away 3 is 2") , 8 0/10 - 4 2/10 = 4 2/10.
3 1/10	"8 take away 4 is 4. L.C.D. is 10 ; 5 go into 10 two times ; 2 + 2 = 4 ; ten go into 10 one time ; 1 + 3 = 4" so wrote 4/10 for 2/5 and 4/10 for 3/10. "You can't subtract 4 from 4 so you borrow one from 4 (remainder from 8 - 4), make it a 3. Make 4 (of 4/10 written for 2/5) a 5. Then 5/10 - 4/10 = 1/10.
3 9/10	Chose 10 as C.D. Wrote 3/10 for 3/10 and 2/10 for 2/5. Borrowed 1 from 8. Made 2/10 into 12/10. Then 7 12/10 - 4 3/10 = 3 9/10.
4	"You can't take 10 from 5, so borrow 1 from 8 and add 5/5 to 2/5" for 7 7/5. Then "3 from 7 is 4." Did not proceed further "because you can't take 10 from 5."
9/	Wrote 8 2/5 as 7 12/5 ; "12 take away 3 is 9 ; 5 can't take away 10, I am stuck on that answer." Stopped.

7 1/2 - 4 1/4* = ___ (Four Schools)

Answers	
3 3/4	"2 and 4 will go into 4" so C.D. = 4. Said "2 x 2 = 4 ; 2 x 1 = 2 ; (so 1/2 became 2/4). Add your 1 (of 1/4) for 3/4. 4 from 7 is 3."
2 0/9	"7 wholes take away 4 wholes would be 3 wholes · 2 won't take away 4, so have to borrow 1 from 7 and make it a 6 ; 6 take away 4 is 2 ; the 2 (of 1/2) would turn into a 12 ; 12 take away 4 would be 9, and 1 take away 1 would be 0." Answer 2 0/9.
3 0/	"7 take away 4 leaves 3 ; 1 take away 1 leaves 0," couldn't proceed further.
3 2/4	Wrote 1/2 as 2/4 and 1/4 as 4/4 ("because we are working on fourths"). Then 7 2/4 - 4 4/4 = 3 2/4.
2/2	Wrote 7 1/2 as 15/2 and 4 1/4 as 17/4. Then "15 from 17 is 2 ; 2 from 4 is 2".
3 0/2	7 - 4 = 3 ; 1 from 1 = 0 (for numerator), 2 from 4 = 2 (for denominator). (4 pupils)
3 2/8	Chose 8 as C.D. Wrote 1/2 as 2/8 (wrote 8 as denominator then 2 x 1 = 2) ; wrote 1/4 as 4/8 (wrote 8 as denominator : then 4 x 1 = 4) : 7 - 4 = 3 and 4 - 2 = 2 for 3 2/8.
5	"7 - 4 = 3 ; 1 - 1 = 0 · 2 from 4 leaves 2" for 3 0/2 "so just say 5 ; add the 2 and 3."
4/1	"2 subtract 7 is 5, subtract 1 more is 4 (numerator), 4 subtract 4 is 0, and 1 more is 1 (denominator).
8	"2 times 7 is 14, minus 1 is 13" ; "4 times 4 is 16 minus 1 is 15" ; wrote 13 below 7 1/2 and 15 below 4 1/4. Then 13 - 15 = 8 ("13 - 5 is 8--counted fingers--0 - 1 is 0"). Answer 8.
3 1/	"4 from 7 is 3 ; 1/4 and 1/2 will give me?? I can't figure out the bottom digits, for the top 1 from 1 will give me nothing. So I will put the 1 down. Could not proceed further.
133	"1 over 2 leave 1 ; 1 over 4 would be 3 ; 7 from 4 would leave 3."
3 1/4	Wrote 1/2 as 3/4 ("2 goes into 4 one time, 1 + 2 = 3"). Wrote 1/4 as 2/4 ("4 goes into 4 one time, 1 + 1 = 2"). "7 - 4 = 3 ; 3 - 2 = 1 ; bring down 4."
5 1/0	"1 from 1 you cannot take, so go to 7 and borrow a whole 1, that leaves 2" (changed 1 of 1/2 to 2) ; "4 from 2 you cannot take, go over here and borrow a whole 2, and that leaves 5," (the 2 of 1/2 became 4). Then 2/4 - 1/4 = 1/0 "that leaves 5" (from changing 7 to 6 and 6 to 5). Ignored whole number 4 in 4 1/4.

* This exercise was not used in interviews in the last 2 schools.

5/8 - 1/3 = ___

Answers

4/5	"1 from 5 is 4, 3 from 8 is 5." (26 pupils)
1	"You can't subtract 8 (of 5/8) from 5, so make the 5 a 15" ; "you can't get 3 (of 1/3) so you have to make the 1 a 10." Then 15 - 8 = 7 (counted) and 10 - 3 = 7. Answer 7/7 = 1.
3/2	"8 take away 5 leaves 3" (numerator) ; "3 take away 1 leaves 2" (denominator).
4/24	C.D. is 24 because 3 x 8 = 24 and 8 x 3 = 24. Wrote $\overline{24}$ and $\overline{24}$. "Put 1 here and 5 here ; it will be 4/24."
Incomplete	Chose 24 as C.D. Wrote 1/3 as 9/24 ("3 will go into 24 eight times ; 1 + 8 = 9") ; wrote 5/8 as 8/24 ("8 will go into 24 three times, 5 + 3 = 8"). Decided "8 (of 8/24) is too small." Could not proceed further.
4/3	First chose 2 as C.D. "2 goes into 8 four times, 2 goes into 3 one time." Changed to 3 as C.D. Wrote 3 as denominator of answer then 5 - 1 = 4 for numerator.
9/24	Wrote 15/24 for 5/8 and 6/24 for 1/3 (3 can go into 24 six times). Then 15 - 6 = 9.
Incomplete	Wrote 2/24 for 5/8 "8 goes into 24 three times ; 5 take away 3 is 2"). Wrote 7/24 for 1/3 ("3 goes into 24 eight times, 8 take away 1 is 7") ; "That ain't gonna work because you can't take 7 from 2." Stopped.
Incomplete	Wrote 2/24 for 5/8 ("8 goes into 24 three times, 3 take away 5 is 2") ; wrote 7/24 for 1/3 ("3 goes into 24 eight times, 8 take away 1 is 7"). "Can't multiply 7 from 2." Stopped.
4/24	Chose 24 as C.D. Then wrote 5/24 for 5/8 and 1/24 for 1/3. Then 5/24 - 1/24 = 4/24.
1/2	Chose 24 as C.D. Then "1 from 5 is 4" for difference of 4/24. Reduced to 2/4 ("because a half of 4 is 2 and I just know 4 will go into 24") ; 2/4 = 1/2.
4/24	Chose 24 as C.D. Then "5 take away 1 is 4" for 4/24.
7/28	Chose 28 as C.D. because "8 times 3 is 28 and 3 times 8 is 28". Wrote 5/8 as 15/28 and 1/3 as 8/28. Then 15/28 - 8/28 = 7/28.
1/8	Chose 16 as C.D. Wrote 10/16 for 5/8 and 8/16 for 1/3. Apparently thought of 16 as C.D. for 5/8 and 24 as C.D. for 1/3 yet wrote 16 for both. Then 10/16 - 8/16 = 2/16 = 1/8.
1/8	Wrote 1/3 as 6/8. To make denominator of 1/3 same as that of 5/8 added 5 to 3. Then must add 5 to numerator and 1 + 5 = 6. Then 5/8 - 6/8 = 1/8 (6 - 5 = 1).
1/4	Wrote 5/8 as 15/24 and 1/3 as 8/24. Then "8 from 15 is 6" for 6/24 = 1/4.

Subtract 9 2/3*
5 7/8 (Two Schools only)

Answers

3 5/24	Wrote 2/3 as 16/24 and 7/8 as 21/24. Borrowed 1 from 9. Made 16/24 into 26/24. Then 26/24 - 21/24 = 5/24 and 8 - 5 = 3. (8 pupils)
4 2/24	Wrote 6/24 for 2/3 ("3 goes into 24 eight times and 8 take away 2 is 6") ; wrote 4/24 for 7/8 ("8 goes into 24 three times and 3 from 7 is 4") ; then 6/24 - 4/24 = 2/24 and 9 - 5 = 4.
1/8	Wrote 16/24 for 2/3 and 21/24 for 7/8. "I can't subtract 21 from 16 so I borrow 1 from 9 and make it 8." Then wrote 24/24 for 16/24. Then 24/24 - 21/24 = 3/24 = 1/8. Ignored whole numbers.
3 5/24	Chose 24 as C.D. Wrote 2/24 for 2/3 and 7/24 for 7/8. Borrowed 1 from 9 made 2/24 into 12/24. Then 8 12/24 - 5 7/24 = 3 5/24.
3 4/5	Wrote 9 2/3 as 8 11/13 ("take 1 from 9, make it an 8, make 2 a 12, take 1 from 12, make 3 a 13"). Then 11 - 7 = 4 and 13 - 8 = 5 ; 8 - 5 = 3.
Incomplete	Wrote 2/3 as 16/24 and 7/8 as 21/24. "Can't take 16 away from 21, cross out the 9, make it an 8." Didn't know what to do with borrowed one.
Incomplete	Wrote 2/3 as 16/24 and 7/8 as 21/24. "I can't take 21 from 16", wrote 32/48 and 42/48. Gave up.
4 0/24	Wrote 2/3 as 10/24 ("3 goes into 24 eight times and 8 + 2 = 10"). Wrote 10/24 for 2/3 ("8 into 24 three times, and 7 + 3 = 10"). 9 10/24 - 5 10/24 = 4 0/24.
3 5/28	Chose 28 as C.D. because "3 x 8 = 28 and 8 x 3 = 28." Wrote 2/3 as 16/28 and 7/8 as 21/28, borrowed 1 from 9 made it 8. Changed 16/28 into 26/28. Then 8 26/28 - 5 21/28 = 3 5/28.
3 6/24	Wrote 2/3 as 16/24 and 7/8 as 21/24. "You can't take 21/24 from 16/24 so you have to cross out the 9, put an 8 ; 2 and 9 is 11 ; and 11 and 16 is 27." Then 8 27/24 - 5 21/24 = 3 6/24.
3	"Cross out 9 make it 8 and 2 (of 2/3) a 12" ; "cross out 8 make it 7 and 3 (of 2/3) a 13." Then 7 12/13 - 5 7/8 = 2 5/5 (7 - 5 = 2 ; 12 - 7 = 5 ; 13 - 8 = 5) = 3.
3/24	Wrote 2/3 as 16/24 and 7/8 as 21/24. Borrowed 1 from 9 make it 8 ; changed 16/24 into 24/24 (probably thinking 8/8 for 1). Then 24/24 - 21/24 = 3/24. Ignored whole numbers.

* This exercise was included in interviews in last 2 schools only.

Subtract 9 2/3 (continued)
5 7/8

2 4/5	"You can't take a higher fraction from a lower fraction so you have to borrow ; to take a whole out of there (the 9) you need 3 thirds, so you take 2 out of there (the 9) ; would be 9/3 and 9/3 + 2 would be 11/3." So 7 11/3 - 5 7/8 = 2 4/5 (7 - 5 = 2 ; 11 - 7 = 4 ; 8 - 3 = 5).
2 5/5	"Can't take 7 from 2 so you have to borrow, make 9 an 8, and 2 (of 2/3) a 12. Can't take 8 from 3 so you have to borrow, cross out 8 and put down 7, make 3 (of 2/3) a 13." Then 7 12/13 - 5 7/8 = 2 5/5 (7 - 5 = 2 ; 12 - 7 = 5 ; 13 - 8 = 5).
2 5/5	"7 into 2 won't go, so cross out 9 and make it 8, that's 12 (made 2 of 2/3 into 12) ; 7 from 12 is 5" ; "8 into 3 won't go so cross out 8 and make it a 7 ; make this 13 (3 of 2/3)." Then 7 12/13 - 5 7/8 = 2 5/5.
4 5/5	"9 take away 5 is 4 ; 2 take away 7 is 5 ; 3 take away 8 is 5."
Incomplete	Wrote 2/3 as 16/24 and 7/8 as 21/24. "Now if I borrow 1 from 9 and put it on that (16 of 16/24) I'd get 17, but I still couldn't do it."

2/3 x 3/5 = ___

Answers

9/10	"You multiply across." 3 x 3 = 9 ; 5 x 2 = 10.
1 1/	"When you multiply, you divide so you say 2 from 3 is 1, it will leave 1." Confused, could go no further.
6 0/15	Wrote 10/15 for 2/3 and 9/15 for 3/5. Then 9 x 10 = 90 over 15. Divided 15 into 90, got 6 and remainder 0. Wrote as 6 0/15.
90/15	Wrote 10/15 for 2/3 and 9/15 for 3/5. Then 9 x 10 = 90 over 15. (27 pupils)
15	Wrote 10/15 for 2/3 and 9/15 for 3/5. Then 9 x 10 = 90 over 15 ; 90/15 = 45/3 = 15.
10/9	2 x 5 = 10 and 3 x 3 = 9. (4 pupils)
15	2 x 3 (of 2/3) = 6 ; 3 x 5 (of 3/5) = 15.
615	2 x 3 = 6 ; 3 x 5 = 15. (2 pupils)
2/15	Chose 15 as C.D. Wrote 2/15 for 2/3 ("15 goes into 3 five times, 5 will go into 2 two times"). Wrote 1/15 for 3/5("15 goes into 5 three times, 3 will go into 3 one time"). Then with 2/15 above 1/15, "2 x 1 = 2, answer 2/15."

2/3 x 3/5 = ___ (continued)

6/15*	2/3 x 3/5 = 2/15 x 3/15 = 6/15. (2 pupils) Answer was accidentally correct.
90	2 x 3 = 6 ; 3 x 5 = 15 ; 6 x 15 = 90.
3/5	2 x 3 = 6 and 3 x 5 = 15 for 6/15 reduced incorrectly to 3/5.
1 4/15	Wrote 10/15 for 2/3 and 9/15 for 3/5. "Add 10 and 9, leave 19 for 19/15 ; that'll be 16, 17, 18, 19, or 1 4/15."
95/19	Wrote 10/15 for 2/3 and 9/15 for 3/5 ; wrote 19 above 15 to multiply : "9 x 5 = 45, carry 4 ; 1 x 5 = 5, + 4 = 9 ; 9 x 1 = 9 and 1 x 1 = 1, makes that 19." Wrote 95/19.
10/9	Wrote reciprocal of 3/5 and multiplied 2/3 x 5/3 = 10/9. (2 pupils)
6/15	2 x 3 = 6 (from 2/3), 3 x 5 = 15 (from 3/5). Answer was accidentally correct.
5/5	"Both 3 and 5 are prime, so 3 and 2 more is 5." The answer is 5/5.
46/15	Chose 15 as C.D. Wrote 7/15 for 2/3 ("3 goes into 15 five times, 5 + 2 = 7") ; wrote 6/15 for 3/5 ("5 goes into 15 three times, 3 + 3 = 6"). Then 7/15 + 6/15 ; 6 x 7 is 46 (counted 18, 24, 28, 34, 46) so answer is 46/15.
incomplete	Wrote 2/3 as 10/15 and 3/5 as 9/15. First thought answer is 1/15 by subtraction, then said "I would like to multiply 9 by 10 but I know that's wrong." Gave up.
100	"2 x 5 = 10, put down 0 and carry 1" ; 3 x 3 = 9, + 1 = 10. Answer 100.
6/5	Wrote 2/3 x 3/5 = 6/5. "5 and 2 won't go, 3 and 3 go only 1 time." Crossed out 3 of 2/3 and made it 1. Then 2 x 3 = 6 and 1 x 5 = 5.
2 2/5	Wrote 2/3 as 4/5. ("Denominators must be same, add 2 to 3 and get 5. Must add same to numerator so 2 + 2 = 4"). Then 4/5 x 3/5 = 12/5 = 2 2/5.")
1/2	"2 goes into 5 with 1 left." Wrote 2[1] below 3/5. "3 can't go into 3," wrote 1 in numerator and 2 in denominator. "Now 1/2 times something but I don't know."
2/3	2/3 x 3/5 = 6/15. "To reduce, divide by 3/3 ; 6 goes into 3 two times ; 15 goes into 3 three times so that'll be 2/3."

---

* This method will yield correct answers if C.D. is product of denominators.

2 1/2 x 6 = ___

Answers	
3/6	"I have 2 whole pies and a half of a pie. I take 6 slices from it, that will be 3 over 6."
12 1/2	6 x 2 = 12, affixed 1/2. (27 pupils)
1 5/2	Rewrote vertically as 2 1/2 above 6/1. "Cannot take 6 from 1 so borrow 1 from 2, make it 1, and make 11/2" (from 1/2). Then 1 11/2 - 6/2 = 1 5/2.
140/2	Wrote 2 1/2 as 5/2. Then 5/2 = 20/2 and 6/1 = 12/2. 12 x 20 = 140.
2 6/2	Bring over 2 ; 1 x 6 = 6 (numerator), bring over 2 "but that's a top heavy number."
12	Wrote 4/2 for 2 1/2 (2 x 1 = 2 ; 4 x 1 = 4) and 6/1 for 6. Then 4/2 x 6/1 = 24/2 = 12.
2 12/2	Wrote 2 1/2 above 6/1. Then 1/2 for 1/2 and 12/2 for 6/1. Then 2 1/2 x 12/2 = 2 12/2 ("12 x 1 = 12 and 2 is C.D.").
17	Wrote 5/2 for 2 1/2. Then 5/2 x 6 = 17 ("6 x 2 = 12, + 5 = 17").
12 5/2	Wrote 2 1/2 = 5/2 above 6 (6 immediately below whole number 2 of 2 1/2). Then "6 x 2 = 12, bring down 5/2."
30	"2 x 2 is 4, + 1 is 5 ; 5 x 6 = 30." (2 pupils)
4 1/6	Wrote 2 1/2 x 1/6 with 1/6 under the 2 1/2. "When we have a whole number, we put a 1 over it. 2 times 2 is 4 ; times 1 is 4." Wrote 4 and added 1/6 for 4 1/6.
2/0	"2 x 1 (of 1/2) = 2", wrote 2 in numerator "and that would be 0" (denominator) since there was no number after 6.
8 1/2	6 x 2 = 8, affixed 1/2. (2 pupils)
60/2	Wrote 5/2 for 2 1/2 and 12/2 for 6. Then 5/2 x 12/2 = 60/2.
10 2/4	"Change 6 to a fraction, make it 5 1/2" (take 1/2 away from 6). Then 1 x 1 = 2 and 2 x 2 = 4", wrote 2/4 ; 5 x 2 = 10.
12 3/1	"6 x 2 = 12 ; 6 x 1 = 6, over 2." Wrote 12 6/2. "2 goes into 6 three times ; 2 goes into 2 one time, answer 12 and 3 over 1."
2 6/12	"Bring over the 2, 6 x 1 = 6 ; 6 x 2 = 12. Answer 2 6/12."
24	"2 x 2 = 4 times 1 is 4." Wrote 4 below 2 1/2. Then 4 x 6 = 24.
7	Wrote 5/2 for 2 1/2 and 6/1 for 6, then 5/2 x 6/1 = 30/2. In reducing "2 x 10 = 20 with 10 left over." This gave 2 10/2, and 10/2 = 5 so 2 + 5 = 7.

2 1/2 x 6 = __ (continued)

3	Wrote 2 1/2 above 6/6 ("6 can be changed to an improper fraction so it will be 6/6"). Wrote 2 3/6 for 2 1/2 above 6/6 ; then "6 + 3 (counted) = 9 bring down your 2." Wrote 2 9/6 ; then counted "6, 7, 8, 9, leaves 3." Wrote answer of 3.
6	2/2 x 6/1 = 12/2 = 6 ; then 5/2 x 6/1 = 30/2 = 15 ; "I don't know which one is right, I just did it two ways."
5	Wrote 2 1/2 x 6/1, multiplied 1/2 x 6/1 = 6/2. Wrote this as 3 then "3 + 2 = 5." (2 pupils)
10 1/2	Wrote 2 1/2 for 2 1/2 above 5 2/2 for 6. Then 5 x 2 = 10 and 1/2 x 2/2 = 2/4 for 10 2/4 = 10 1/2.
2/24	Wrote 2/4 for 1/2 and 1/6 for 6. Then 2/4 x 1/6 = 2/24.
8 1/2	Wrote 6 x 5/2 said "five halves times 6 is 12 and 12 and 5 is 17, so 17/2 = 8 1/2."
2 36/12	Arranged work like this. Chose 12 for C.D. ("6 is 2 x 3 and 2 is prime, so that'll be 2 x 2 = 4 and 4 x 3 = 12"). Then "6 goes into 12 six times." Wrote 6/12 by the 6. "You automatically put a one there" (to left of 6). Then "2 goes into 12 six times and 6 x 1 = 6." Wrote 6/12 by 2 1/2. "6 x 6 = 36 over 12 and 2 x 1 = 2, so it's gonna be 2 36/12." $2\frac{1}{2}$ $\frac{6}{12}$ 16 $\frac{6}{12}$ 2 36/12
5/12	Wrote 2 1/2 as 5/2 and 6 as 1/6. Then 5/2 x 1/6 = 5/12. (2 pupils)
2 7/2	Wrote 2 1/2 above 6/1. "6 + 1 = 7 and 2 x 1 = 2." Wrote 7/2 then affixed the 2 for 2 7/2.
12 2/3	"2 x 6 = 12 ; 6 x 2 = 12, and 6 x 1 = 6." Wrote 12 6/12 then reduced 6/12 to 2/3 ("2 can go into 6 and 3 can go into 12").
8 1/2	"2 into 6 would be 8 and put down 1/2."
11	Wrote 2 1/2 above 5 2/2 ("I am going to borrow 1 from 6 and make it 5 and 2 twos"). Then 5 x 2 = 10 and 1 x 2 = 2 so 10 2/2 = 11.
120	Wrote 6 below 2 1/2. "0 times 1/2 = 0. There is nothing under 1/2 so multiply by 0, 6 x 2 = 12." Answer 120.
60/2	Wrote 2 1/2 as 5/2 and 6 as 12/2. Then 5/2 x 12/2 = 60/2.
15/2	Wrote 2 1/2 as 5/2 and 6 as 6/1. Then 5/2 x 6/1 = 30/2 reduced to 15/2 ("2 goes into 30 fifteen times and the denominator is 2").
12 1/4	"2 x 6 = 12 ; 1/2 x 1/2 = 1/4" (since there was nothing beside the 6 he mentally affixed 1/2), making the exercise 2 1/2 x 6 1/2.

5 1/2 x 3/4 = ___

<u>Answers</u>

5 3/8	"1 x 3 = 3 ; 2 x 4 = 8, bring over 5." (24 pupils)
15 1/4	Wrote as 15 3/4 ÷ 3/4 = 15 1/4. "Always keep whole number (which he wrote as 15) and denominator (4)." Then "3 into 3 goes 1."
77	Wrote 11 for 5 1/2 ("5 x 2 = 10, + 1 = 11). Then "11 x 3 = 33 ; 11 x 4 = 44 ; 33 + 44 = 77."
5.24	Considered cancelling "but that wouldn't work because you need another whole number over here (3/4) or not one over here (5 1/2) ; you need 2 whole numbers or none at all if you are to cancel." Then "1 x 4 = 4" (1 from 1/2 and 4 from 3/4). Wrote 4 ; "2 x 3 = 6" (2 of 1/2 and 3 of 3/4), wrote 6 by 4 like this 4.6 ; "4 x 6 = 24 bring over 5 and answer is 5 twenty-fourths" written 5.24.
5 5/4	Rewrote vertically. Then 2/4 for 1/2 and 3/4 for 3/4 "3 x 2 is 5, that's 5/4, bring down 5."
7 1/3	Wrote 11/2 for 5 1/2 and reciprocal of 3/4. Then 11/2 x 4/3 = 44/6 = 7 2/6 = 7 1/3. (2 pupils)
5 4/6	"4 x 1 = 4 ; 2 x 3 = 6, bring over 5."
66/4	Chose 4 as C.D. Wrote 5 1/2 as 22/4 and 3/4 as 3/4. Then 22/4 x 3/4 = 66/4.
4 1/2	Arranged work this way 3/4 x 5/1 = 15/4 = 3 3/4 1/2 x 3/4 = 3/4 = 3/4 3 6/4 = 4 1/2
30 3/8	Couldn't do it with only one whole number so interviewer made exercise 5 1/2 x 6 3/4. Then "1 x 3 = 3 ; 2 x 4 = 8 ; 5 x 6 = 30."
16 1/2	Wrote 5 1/2 as 22/4. Then 22/4 x 3/4 = 66/4 = 16 1/2.
3/4	Chose 4 for C.D. Wrote $\frac{\ }{4}$ and $\frac{\ }{4}$. Then wrote 5 1/4 x 3/4 = 3/4 ("will have a 5 here, a 1 here, and a 3 here"). "1 x 3 = 3." Wrote 3/4. "I don't know how to multiply by 5 ; I'm not sure whether to multiply by 4, or 3, or either one, 3/4 of 5 is 4, but I don't know what to do with it." Left answer 3/4.
5 4/8	"3 x 1 = 4 ; 4 x 2 = 8 ; bring over 5."
5 2/3	"1 x 4 = 4 ; 2 x 3 = 6 ; 4/6 = 2/3." Affix 5 for 5 2/3.
30/40	"2 x 5 is 10 ; 10 x 1 is 10." Wrote 10 under 5 1/2. Then "10 x 3 = 30 ; 10 x 4 = 40." Answer 30/40.
8 0/8	Wrote 4/8 for 1/2 and 6/8 for 3/4. Then 4/8 x 6/8 = 24/8. Reduced to 3 0/8, added 5 for 8 0/8.

5 1/2 x 3/4 = ___ (continued)

5 30/150	Wrote 5 1/2 above 3/4. Then 1/2 = 6/12 and 3/4 = 9/12. 6 + 9 = 15 and 15 x 12 = ? Then 15 x 12 wrote 12. "5 x 2 = 10, write 0 carry 1 ; 1 x 2 = 2, 15 + 1 = 3" for 30 (numerator). "Then I write a 0 (in denominator) and 1 x 5 is 5, 1 x 1 is 1." Wrote 150 in denominator and brought down 5 for 5 30/150.
5 3/2	Wrote 5 2/4 for 5 1/2 above 3/4. Then 2 x 3 = 6 over 4. Affixed 5 for 5 6/4 = 5 3/2.
22/3	Wrote 11/2 for 5 1/2 and reciprocal of 3/4. Then 11/2 x 4/3 = 22/3 (cancelled 2 into 4).
11 3/4	"5 x 2 = 10, + 1 = 11, so 11 3/4." Then 4 x 11 = 44, + 3 = 47, or 47/4 = 11 3/4.
15/40	Wrote 5/10 for 1/2. Then 5/10 x 3/4 = 15/40. Ignored whole number 5.
5 2/4	Wrote 2/4 for 1/2 and 3/4 for 3/4. Then 2 x 3 = 6 for 6/4 = 1 2/4 ; "that'll equal 5 2/4."
5 6/4	Wrote 2/4 for 1/2 and 3/4 for 3/4. Then 2 x 3 = 6 for 6/4 ; bring over your 5."
5 3/4	Chose 4 as C.D. Wrote 1/2 as 1/4 and 3/4 as 3/4. "3 x 1 is 3." Wrote 3/4 and affixed 5.
6 2/4	Wrote 2/4 for 1/2 and 3/4 for 3/4 ; "3 x 2 = 6." Wrote 6/4, affixed 5 for 5 6/4 = 6 2/4. (2 pupils)
5 3/4	"First it would be 5, then 1 x 3 is 3 and C.D. is 4 so 5 3/4." (2 pupils)
4 3/8	Wrote 5 1/2 above 3/4. "Have to find the denominator, so you borrow 1 from 5, make that a 4, add it on to the 3/4, makes that 1 3/4 and 1 times 4 is 4." Then 1/2 x 3/4 = 3/8 ("3 x 1 = 3 and 4 x 2 = 8").
5 3/4	"This is 4 (wrote as denominator of fraction in answer), and this is 5" (wrote as whole number of answer), "1 x 3 is 3" so 5 3/4.
26 7/8	Wrote 5 1/2 as 11/2 and 3/4 as 3/4. Then 11/2 x 3/4 = 33/8. Counted from 8 to 33 on fingers for 26 then from 26 to 33 and got "7 left" so 26 7/8 answer.
16 1/2	Chose 4 as C.D. Wrote 15 2/4 x 3/4 = 15 6/4 = 16 2/4 = 16 1/2 (2/4 x 3/4 = 6/4 and 3 x 5 = 15 for 15 6/4).

$9/10 \div 3/10 =$ ____

Answers

3/10	"3 into 9 goes 3 times, bring over the 10." (61 pupils)
27/10	"When you divide you multiply, that would give you 27/10" (3 x 9 = 27).
3/0	"10 divided by 10 is 0 ; 9 divided by 3 is 3."
3/20	Wrote 18/20 for 9/10 and 6/20 for 3/10. Then 18/20 ÷ 6/20 = 3/20 ("6 into 18 goes 3 bring down 20").
4 4/9	Wrote vertically 1 1/9 for 9/10 and 3 1/3 for 3/10. Then 3 + 1 = 4 and 1/9 + 3/9 = 4/9.
3/0	"9 into 3 leaves 3 (numerator), 10 into 10 leaves 0" (denominator). (3 pupils)
6/10	"9 divided by 3 = 6" so 6/10.
27/10	"The denominator would be 10 ; 3 x 9 = 27 ; and 27 would be numerator." (2 pupils)
24	"9 over 10 go 1 time" (1 left over): wrote 1 1 "3 over 10 goes 3 times with 1 left over." 1 3 Wrote 1 and 3 under the ones as here. Then added 1 + 1 = 2 and 1 + 3 = 4 for 24.
6	"9 divided by 3 = 6 and 10 divided by 10 = 1," so 6/1 = 6.
1/3	3 into 9 is 3 and 10 into 10 is 1. First wrote answer as 3/1 then rewrote as 1/3.
3 2/3	Divided 3 into 9 this way 3/9 got 3 R2. Wrote as 3 2/3.
Incomplete	"9 won't go into 3 ; 3 goes into 9 three times and 3 is prime and 9 x 3 = 27, that ain't gonna work." Gave up.
12/20	Wrote 9/10 above 3/10 "that'll be 12/10." Uncertain what to do next. Wrote 2 under 12/10. Then wrote 12/10 x 2/1 = 12/10 x 1/2 = 12/20.
0/10	"C.D. would be 10, and 9 can't go into 3 so answer is 0/10." (2 pupils)
6/0	"9 divided by 3 is 6, and 10 divided by 10 is 0." (3 pupils)
36/20	"3 x 9 is 36, and 10 times 10 is 20."
1/3	"Turn it over" (9/10). Wrote 10/9 x 3/10. Then cancelled 3 into 9 and 10 into 10. Product 1/3.
30	"9 divided by 3 is 3 and 10 divided by 10 is 0."

15 3/4 ÷ 3/4 = ____

Answers	
15 0/4	"3 divided by 3 is 0 and C.D. is 4 so it would be 15 0/4." (2 pupils)
15 3/4	"Bring over 15 ; 4 divided by 4 = 4 and 3 ÷ 3 = 3. Answer 15 3/4."
15 1/4	"3 ÷ 3 = 1, bring over 15, answer is 15 1/4." (10 pupils)
16	"3/4 ÷ 3/4 = 1, bring over 15 and 15 + 1 = 16" or "3 ÷ 3 = 1 ; 4 ÷ 4 = 1 ; 15 1/1 = 16." (10 pupils)
7 2/3	Wrote 15 3/4 as 23/4, and reciprocal of 3/4. Then 23/4 x 4/3 = 23/3 (cancelled) = 7 2/3.
21/4	Wrote 15 3/4 as 63/4. Then 63/4 ÷ 3/4 = 21/4. (3 pupils)
20/1	Wrote 15 3/4 as 60/4. Then 60/4 ÷ 3/4 = 60/4 x 4/3 = 20/1 (cancelled). (2 pupils)
4/189	Wrote 15 3/4 as 63/1 ("15 times 4 is 60, + 3 = 63, you put whole numbers over 1"). Then 63/1 ÷ 3/4 = 1/63 x 4/3 = 4/189.
15	Wrote 15 3/4 x 4/3 (cancelled) = 15 1/1 = 15.
15 0/0	"3 into 3 = 0 ; 4 into 4 = 0," wrote 15 0/0.
15 1/1	"3 goes into 3 one time, 4 goes into 4 one time" bring over 15. (5 pupils)
16 1/2	"3/4 + 3/4 = 6/4" bring over 15 for 15 6/4 = 16 2/4 = 16 1/2.
15	"3/4 from 3/4 is 0, put down 15." (2 pupils)
12	"3 divided by 3 is 1 bring over 4" for 3/4 ; 3/4 of 15 is 12."
1/4	"Numerator will stay the same if same number is on bottom, and then 3 will go into 3 one time" so 1/4.
1/4	"3 go into 3 one time ; 4 can't go into 4, so you regroup a whole 4 from 15, that leaves 14." Then 14 3/4 ÷ 3/4 = 1/4 (the 4 of 14 3/4 becomes 8 and 4 from 8 leaves 4, answer 1/4").
15 9/4	"Bring over 15 ; 3 x 3 = 9 ; bring over 4."
21 R1	Wrote 15 3/4 as 64 ("4 x 5 = 24 ; 4 x 10 = 40 ; 24 + 40 = 64"). Divided 64 by 3 got 21 R1.
15	"Keep 15 ; 3/4 will go into 3/4 but it won't equal anything, so answer is 15."
5 1/4	"3/4 into 3/4 goes 1 time, that gives 1/4 ; 3/4 into 15 ?? divide 3 in 15, that's 5." Answer 5 1/4.
16	Wrote 15 3/4 as 63/4 ; then "3 goes into 60 fifteen times, and 3 goes into 3 one time, so answer is 16."
7/4	Wrote 63/4 for 15 3/4. Then 63 ÷ 3 = 21 and 21 ÷ 3 = 7. Used C.D. of 4 for 7/4.
20	Wrote 15 3/4 as 60/4 ; then 60/4 ÷ 3/4 = 60/4 x 4/3 = 20 (cancelled) (in multiplying 15 x 4 said "10 x 4 = 40, 11 x 4 = 44 ; 12 x 4 = 48 ; 13 x 4 = 52 ; 14 x 4 = 56 ; 15 x 4 = 60").

15 3/4 ÷ 3/4 = ____ (continued)

20/4	Wrote 15 3/4 as 60/4 so it's 60/4 ÷ 3/4 · 3 into 60 goes 20 times so answer is 20/4. (2 pupils)
11 R13	Wrote 63/4 for 15 3/4. Then 63/4 3/4 189/16 = 11 R13. ("4 won't go into 63 so 63 x 3 = 189 ; 4 x 4 = 16.") Divided 16 into 189, answer 11 R13.
0/1	Wrote 63/4 for 15 3/4. Then 63/4 ÷ 3/4 = 0/1. ("3 won't go into 63"), wrote 0 for numerator, "4 into 4 goes 1" wrote 1 for denominator.
151	"15 divided into 15 is 15 ; 3/4 divided by 3/4 is 1 time." Wrote 151.
3 3/4	Wrote 45/4 for 15 3/4. Then 45/4 ÷ 3/4 = 15/4 = 3 3/4.
5 3/4	"3 (of second 3/4) into 15 is 5 and bring down 3/4."
0 1/4	"How many 3's in 3 ; there's 1 three in 3 : it's the same number so it stays 4" wrote 1/4. Then "0 divided by 15 is 0" wrote 0 1/4.

6 9/10 ÷ 3 = ____

Answers

2 9/10	"6 into 3 will go 2 ; and 9/10," or "6 whole ÷ 3 whole is 2. You ain't got nothing to divide 9/10 by so put down your 9/10." (25 pupils)
2 3/2	First said "can't do it because there is no fraction behind 3." Interviewer made exercise 6 9/10 ÷ 3 3/5. Then "3 goes into 9 three ; 5 goes into 10 two, 3 goes into 6 two" ; answer 2 3/2.
2 3/3	"3 go into 9 three times ; 3 go into 10 three times and 1 left ; 3 goes into 6 two times."
6 27/30	"3 x 9 = 27 ; 3 x 10 = 30, bring over 6."
6 3/10	Wrote 6 9/10 ÷ 3/1 ; "3 goes into 9 three times ; 1 goes into 10 ten times, bring over the 6." (3 pupils)
20 7/10	Wrote 69/10 x 3/1 = 207/10 = 20 7/10.
2/30	Wrote 6 27/30 for 6 9/10 and 10/30 for 3. Then 6 27/30 ÷ 10/30 = 2/30 R 7/30. "I got messed up." Stopped.
Incomplete	Wrote aside 60 x 9 = 540. Then 540/10 ÷ 3/1 = ? Could not finish.
6 3/10	Rewrote as 6 9/10 ÷ 3/10 (because common denominator is 10) "9 divided by 3 is 3, put over C.D. of 10, bring 6 over."
2 7/10	Wrote 69/10 ÷ 3/1 = 207/10 = 2 7/10.

6 9/10 ÷ 3 = ___ (continued)

1/207	Wrote 69/1 (10 x 6 = 60, + 9 = 69, put it over 1 "because it is a whole number"). Then 69/1 ÷ 3/1 ; "that will be 1/69 x 1/3 = 1/207."
207/10	Wrote 69/10 ÷ 3 = 69/10 x 3/1 = 207/10.
23	"6 x 10 = 60, + 9 = 69 ; 69 ÷ 3 = 23."
135/30	Chose 30 as C.D. Wrote 27/30 for 9/10 and 10/30 for 3 ; "put a 1 under the 6 automatically" ; "27 is 3 to 3rd power and 30 is 2 x 3 x 5 ; so it's gonna be 3 threes is 9 ; and 2 times 3 is 6 ; 6 x 9 is 27 ; and 27 x 5 is 135 ; 6 won't go into 1 so answer is 135/30."
1 2/30	"You break the 3 down and make it 2 3/3" ; wrote 27/30 for 9/10 and 30/30 for 3/3. "Borrow 1 from 6" (of 6 9/10), made 27/30 into 37/30 ; then 30 into 37 = 1 7/30. Then made 2 of 2 3/3 into 3 because of 1 of 1 7/30. Then "3 goes into 5 (left from borrowing 1 from 6) one time with 2/30 left over."
2	"3 goes into 6 two times ; 3 does not go into 9/10 so the answer would be 2."
6 9/30	Wrote 9/10 x 1/3 = 9/30 ; brought over 6.
2 R9	Wrote 69/10 for 6 9/10 and 30/10 for 3 to get a C.D. 30 into 69 = 2 R9.
3 9/10	Wrote 69/10 ÷ 30/10 "69 into 30 would go 3 times with 9/10 left over."
19 R8	Wrote 69/10 for 6 9/10 and 3/1 for 3. Like this: 69 3 Then 3 x 69 for 198. 198/10 = 19 R8. 10 1 And 10 x 1 for 10.
1/0	Wrote 69/10 ÷ 1/3, "1 into 69 goes 1 time ; 3 won't go into 10, so it's 0 ; answer 1/0."
2 1/1	"6 ÷ 3 is 2 ; now 9/10 ÷ 9/10 ; 9 divided by 9 is 1 ; 10 divided by 10 is 1 ; answer 2 1/1."
23	Wrote 69/10 x 1/3 = 23 (cancelled 3 into 69).

7/8 ÷ 2/3 = ____

Answers

3/2	"2 goes into 7 three times, you'll have a remainder, 3 goes into 8 two times, but you still have a remainder." Wrote 3/2 as answer.
14/24	"7 x 2 = 14 ; 8 x 3 = 24, that's not right, you don't multiply the bottom number," could go no further.
1/24	Wrote 21/24 for 7/8 and 16/24 for 2/3. "Now divide 16 into 21 goes 1 time." Answer 1/24.
3 1/24 / 24	Chose 24 as C.D. Wrote 7/24 for 7/8 and 2/24 for 2/3. Then 7/24 ÷ 2/24 = 3 1/24 / 24 (7 ÷ 2 = 3 1/24)

7/8 ÷ 2/3 = ___ (continued)

1	Wrote 7/8 ÷ 2/3. Then cancelled: $\frac{7}{\cancel{8}\,4} \div \frac{\cancel{2}^{1}}{3} = 7/7 = 1$
9	First thought to divide 7 by 2 but it wasn't even ; then 7 x 8 = 56 and 2 x 3 = 6. Wrote 56 ÷ 6 = 9 (6 x 7 = 42, + 9 = 50, + 9 = 56).
34/22	"7 ÷ 2 goes 3 with 1 over." Wrote 3 in numerator of answer ; didn't know what to do with remainder, decided to put it with 2 of 2/3 making it 12/3. Then "3 into 12 goes 4." Wrote 4 with 3 making numerator 34. "3 goes into 8 two times with 2 left over." Wrote 22 as denominator.
2/2	"2 divide 7 is 2 ; 2 divide 8 is 2."
3/2	"2 goes into 7 three times ; 3 goes into 8 two times." (4 pupils)
1 5/24	Wrote 21/24 for 7/8 and 16/24 for 2/3. Then 21/24 ÷ 16/24 = 1 5/24. (12 pupils)
3/5	"3 into 7 go 3 times : 3 into 8 go 5 times."
1/24	Chose 24 as C.D. Wrote 10/24 for 7/8 ("8 goes into 24 three times, 7 + 3 = 10") ; wrote 10/24 for 2/3 ("3 goes into 24 eight times, 8 + 2 = 10"). "10 goes into 10 one time, bring down 24."
3/2 R3	"2 go into 7 three times ; 3 go into 8 two times : so you get 3 left" (1 left from 7 ÷ 2 and 2 left from 8 ÷ 3). (2 pupils)
Incomplete	Wrote 21/24 for 7/8 and 16/24 for 2/3. Then 21/24 ÷ 16/24 = said "it won't work because 16 won't go into 21." Couldn't go further.
$\frac{1\ 5/16}{24}$	Wrote 21/24 for 7/8 and 16/24 for 2/3. Then 21 ÷ 16 = 1 5/16. Was not satisfied with this answer. Then uncertainly wrote answer as here.
3 R1	First wrote 21/24 for 7/8 and 16/24 for 2/3. Then added for 37/24. Decided this was wrong. Then said "2 goes into 7 three times with 1 left over." Wrote 3 R1.
1/24	Wrote 21/24 for 7/8 and 16/24 for 2/3. Then 21 ÷ 16 = 1 R5. Decided final answer must be 1/24 ("24 is the C.D., the 1 is from 1 R5 and 5 can't be any fraction").
4 1/2	7 ÷ 2 = 3 R1 ; 8 ÷ 3 = 2 R2. Wrote 3/2 = 1 1/2. Then R of 1 and R of 2 added to 1 gave 4 1/2.
1 R5	Wrote 21/24 for 7/8 and 16/24 for 2/3. Then 21 ÷ 16 = 1 R5. (2 pupils)
7/12	Wrote 7/8 ÷ 2/3 = 7/8 x 2/3 = 14/24 = 7/12.
1 1/2	"2 goes into 7 three times ; 3 go into 8 two times and 3/2 = 1 1/2." (2 pupils)

7/8 ÷ 2/3 = ____ (continued)

3/5 — "2 goes into 7 three times R1 ; 3 go into 8 two times R2." Then wrote 2/2 1/3 = 3/5. (The 1 of 1/3 is remainder of 7 ÷ 2 ; the 3 of 1/3 is R of 7 ÷ 2 + R of 8 ÷ 3) ; 2/2 + 1/3 = 3/5.

1/2 — "3 divided by 8, two times · 2 divided by 7 equals 3 with R1." Wrote

$$\begin{array}{r} 2 \\ 3\overline{)8} \\ 6 \\ \hline 2 \end{array} \qquad \begin{array}{r} 3 \\ 2\overline{)7} \\ 6 \\ \hline 1 \end{array}$$

Made 1/2 from remainders.

1 2/24 — Wrote 3/24 for 7/8 ("8 goes into 24 three times, and 3 goes into 7 three times and 1 left over"). Wrote 4/24 for 2/3 ("3 goes into 24 eight times and 8 ÷ 2 = 4"). Then 3/24 ÷ 4/24 = 1 1/24 ("3 goes into 4 one time with 1/24 left over.")

37/48 — Wrote 21/24 for 7/8 and 16/24 for 2/3. Added for 37/24. Placed 2 under 37/24. Then wrote 37/24 x 1/2 = 37/48.

0 — Wrote 21/24 for 7/8 and 16/24 for 2/3 ; then "21 divided by 16 is 0 because 21 doesn't go into 16."

1 2/3 — First wrote 21/24 for 7/8 and 16/24 for 2/3. Decided this was wrong. Then "7 divided by 2 is 3 ; 3 x 2 is 6 and 7 - 6 is 1 ; 3 go into 8 two times ; I put the 2 up there (for numerator) and leave the 3."

1 3/24 — Chose 24 as C.D. "Bring down your 2 and your 7" ; wrote 7/24 for 7/8 and 2/24 for 2/3 ; "2 goes into 7 three times and 1 left over ; it'll be 1 and 3/24."

4/21 — "Can't divide 2 by 7." Wrote 8/7 ÷ 2/3 ; "2 divided by 8 is 4 ; 3 x 7 = 21."

3/24 — Chose 24 as C.D. "2 goes into 7 three times, answer 3/24."

3 R1 — "2 goes into 7 three times with one left over." "These are the kind I don't understand, when one denominator is even and one is odd."

14/24 — "2 x 7 = 14 and 3 x 8 is 24."

0/24 — "7 can't go into 2 and 24 is C.D. so answer is 0/24."

16/21 — Wrote 8/7 x 2/3 = 16/21.

3 1/24 — Chose 24 as C.D. "7 divided by 2 goes 3 times with 1 left over."

7/12 — Wrote 7/8 2/3 cancelled 2 into 8 then 7 x 1 = 7 and 4 x 3 = 12.

3/3 — "2 won't go into 7 but it will go into 6" (2, 4, 6). Wrote 3 for numerator ; "3 won't go into 8 but it will go into 9 three times" (3, 6, 9). Wrote 3 for denominator.

7/8 ÷ 2/3 = ____ (continued)

5 1/24	Wrote 21/24 for 7/8 and 16/24 for 2/3 ; "16 will go into 21 one time with 5 left over, so answer is 5 1/24."
305/206 R2	Divided 2 into 7 and 3 into 8 as below.

```
   305          206 R2
 2/70          3/80            Answer   305  R2
   6             6                      ---
   -             -                      206
   1             2
   0             0
   --            --
   10            20
   10            18
   --            --
                  2
```

1	Added 5 to 3 and 5 to 2 of 2/3 to make 7/8. Then 7/8 ÷ 7/8 ; 7 ÷ 7 = 1 ; 8 ÷ 8 = 1 and 1/1 = 1.
9 1/3	Wrote again 7/8 ÷ 2/3 "the 8 into 2 is 4." Wrote 4 above cancelled 2 and 2 below cancelled 8 "and this is once" crossed out 2 below 8 and wrote 1. Then 7 x 4 = 28 and 1 x 3 = 3 for 28/3 = 9 1/3.
10 R5	Wrote 21/24 for 7/8 and 16/24 for 2/3. Then 21 ÷ 16 = 10 R5

```
      10
  16/21
     16
     --
      5
      0
     --
      5
```

APPENDIX C

All Wrong Answers--Comparison Exercises
Reasons for Choices

Which is larger: ___ 2/3 x 5 ___ or 1 x 5?

"1 x 5 is 5 and 3 x 5 = 15. If I make it out a fraction it will be 2 x 15 which is greater than 5."
"1 x 5 = 5; 2/3 x 5 = 5 2/3 which is greater than 5." (2 pupils)
"1 x 5 = 5 and that is less than 2/3 x 5"
"2/3 x 5 has three numbers while 5 x 1 has only two numbers."
"1 is a whole number and 2 (of 2/3) is just a half of something."
"5 x 2 = 10, and bring down the 0; 5 x 3 = 15; that's 150, and 5 x 1 = 5."
"2/3 x 5/1 = 2/3 x 1/5 = 2/15 and 2/15 is greater than 5."
"2 x 5 is 10, and you get 10/3 which is greater than 5."
"2/3 x 5 = 17 (5 x 3 = 15, + 2 = 17), which is greater than 1 x 5."
"5 x 2 = 10 and 5 x 3 = 15 which is greater than 1 x 5." (3 pupils)
"5 2/3 is greater than 5."
"2/3 x 5 = 10/3 which is greater than 1 x 5." (4 pupils)
"5 x 1 is 5 and 5 x 2/3 would be more." (4 pupils)
"2/3 x 5 would be 10/15 and that would be larger."
"That's 15 there (2/3 x 5), and that just be more."
"2/3 x 5/1 = 15, or something."
"Just a guess."
"2/3 is greater than 1." (2 pupils)
"1 x 5 is 5 and 2/3 x 5 will be a mixed number."
"2 x 3 = 6, and 5 x 6 = 30, and 1 x 5 is only 5."
"2/3 x 5 = 25; 2 x 5 and 3 x 5; and 1 x 5 is only 5."
"2/3 is greater than 5."
"I just felt like picking that one."

Which is larger: ___ 3/2 x 6 or ___ 1 x 6?

"3 don't go into 2."
"1 is greater than 3/2." (7 pupils)
"3/2 is greater than 1." (2 pupils)
"6 x 1 = 6 and 3/2 x 6 = 18/12."
"6 is greater than 3/2 x 6."
"1 x 6 gives you a whole number."
"1 is a whole and 3/2 isn't."
"1 is not a fraction."
"6 is greater than 6 3/2."
"1 x 6 is 6 and 3/2 x 6 = 18/2 and 18/2 is greater than 6."

Which is larger: ___ 3/2 x 6 or ___ 1 x 6? (continued)

"3/2 is a part of 1."
"1 is $\frac{1}{1}$ is a whole and 3/2 is part of a whole."

No reason given for wrong choice.

Which is larger: ___ 17 ÷ 5/8 or ___ 17 ÷ 1?

"5 x 17 = 85 and 85 ÷ 8 = 1; 8 goes into 8 one; 8 goes into 5 no times, then 17 is larger" (10 is less than 17).
"Just a guess."
No reason given for wrong choice.
"5/8 is not a whole number." (2 pupils)
"17 ÷ 1 = 17; 17 ÷ 5/8 goes 3 2/8."
"You have to take 5 into 17 and 8 into 17 and you only take that (17 ÷ 1) one time."
"I don't know how I got it."
"1 is greater than 5/8." (9 pupils)
"5/8 is less than 1." (2 pupils)
"17 ÷ 1 would be 17 and I think 17 would be larger than 5/8 divided by 17."
"That's a whole and that's just 5/8."
"17 ÷ 1 = 17, 17 ÷ 5/8 is less than 17." (2 pupils)
"1 is a whole number and 5/8 doesn't equal a whole number." (3 pupils)
"1 ÷ 17 = 17, and 17 ÷ 5/8 is less than 17."
"5/8 is not as large as 1."
"This is 1 and this is 5/8."
"17 ÷ 5/8 = 16."
Didn't know a reason.
"1 is a whole and 5/8 is a fraction." (2 pupils)
"If you go one time you still have 17."
"5/8 is not 1 yet; it is 3/8 less."

Which is larger: ___ 17 ÷ 5/2 or ___ 17 ÷ 1?

"5/2 is greater than 1." (9 pupils)
"Just a guess."
"5/2 is half of a number."
"5/2 would be 2 1/2 and 2 1/2 is greater than 1." (4 pupils)
"5/2 is a whole with 1 left over, that (1) is a whole with none left over."
"When you work it out, you get 17 x 2 which is greater than 17."
"Divide 2 1/2 in 17 and get 6 and something so 5/2 because you are dividing by a larger number."
"5/2 might go into 17 more than 17 times."
"2 goes into 5 and 1/2 so that'll be 2 1/2 compared to 1."

Which is larger: ___ 17 ÷ 5/2 or ___ 17 ÷ 1?

"Numerator is greater than denominator in 5/2 = 2 1/2 which is greater than 1."
"17 ÷ 5/2 you come out with a mixed number."
"17 ÷ 5/2 would be more after you times it."
"5/2 is less than 1 and the lowest would be larger."
"In 17 ÷ 5/2 you have more to multiply by, that is, 17 x 2/5."
"It would be a fraction and probably would come out a little larger, because with this (17 ÷ 1) you have to add a bunch of decimals and you would come out with decimals."
"1 is greater than 5/2 so when you divide it will be larger."
"You take less out when you divide than when you take one."
"2/2 = 1, this is 3/2 more, so 17 ÷ 5/2 is more than 17 ÷ 1."

Which is larger: ___ 3 9/10 + 7/8 or ___ 3 9/10 + 1?

"You need a C.D. of 40 on left; 3 9/10 + 7/8 = 3 16/40 (9 of 9/10 + 7 of 7/8) and 3 16/40 is greater than 3 9/10."
"7/8 is greater than 1." (8 pupils)
"I just think that it is larger."
"3 is a whole and you have 9/10 of a half."
Reason not given for wrong choice.
Reason not recorded--tape fouled.
"If you got a C.D. The left side would be larger."
"3 9/10 + 1 is just like 4 9/10; 3 9/10 + 7/8 is like you add 7 on 9 and 8 on 10 which is greater than 4 9/10."
"3 9/10 + 7/8 would be 16/80 and 3 9/10 + 1 would be 4."
"3 9/10 + 7/8 = 3 16/18 and 3 9/10 + 1 = 4 9/10 and 3 16/18 is greater than 4 9/10."
"You multiply 9/10 and 7/8; you come out with a higher number."
"7 go into 8 one time with a remainder and this is just 1."
"In 3 9/10 + 7/8 it will be 7 + 3 = 10, and 3 9/10 + 1 is just 3 + 1."
Had no reason for wrong choice.
"Just looks bigger."

Which is larger: ___ 8 5/6 + 7/4 or ___ 8 5/6 + 1?

"1 is greater than 7/4." (12 pupils)
"1 is a whole number and 7/4 is just a fraction of a whole."
"8 + 1 is 9 and 9 5/6 is greater than 8 12/10" (7 + 5 = 12, 6 + 10 = 4).
"8 5/6 + 7/4 won't give more than 9 5/6."
"7/4 is part of 1."
No reason recorded for wrong choice.
"You get all the pie instead of just 7 (pieces) of 4 pies."
"1 is a whole and 7/4 is part of a whole." (5 pupils)

Which is larger: ___ 8 5/6 + 7/4 or ___ 8 5/6 + 1?

"8 5/6 + 7/4 = 8 12/10 = 9 2/10; 8 5/6 + 1 = 9 5/6."
"7/4 doesn't make a whole number and 1 does." (2 pupils)
"8 5/6 + 1 = 9 5/6 and 8 5/6 + 7/4 would only be 8 12/6."
"7 can't go into 4."
"7/4 is not as large as 1."
"7/4 is not even half of one."
"7/4 is not an improper fraction (meaning mixed number)."
"8 + 1 is 9 and 8 5/6 + 7/4 would be 8 and something."
"A whole is greater than 7/4."
"8 5/6 + 1 would be 9 5/6 and 8 5/6 + 7/4 would be 8 7/4."
"On the left you end up with 8 and a fraction, and on the right you get 8 5/6 + 1."
"1 is a whole and 7/4 is still a small piece."
"1 is a whole number which is greater than a fraction."

Which is larger: ___ 10 1/9 - 7/8 or ___ 10 1/9 - 1?

"10 1/9 is on both sides, change 1 to a fraction of 8/8. Then 8/8 is greater than 7/8."
"If you subtract the 1 from the 9 of 10 1/9 you get 10 1/8."
"If you subtract 8 of 7/8 from the 9 of 10 1/9 you get 10 1/1 which is greater than 10 1/8."
"1 is more than 7/8." (4 pupils)
Reason not recorded.
"7/8 ain't a whole number."
"I don't know why."
"10 1/9 cancels 10 1/9 and a whole is greater than 7/8."
"It is better to subtract 1 whole than to subtract the fraction."
"10 1/9 - 1 is greater than 10 1/9 - 7/8."
"This is a whole and that's just 7/8."
"You subtract 1 from 10 1/9 you get 9 1/9, you subtract 7/8 from 10 1/9, you get 7 from 1 is nothing and 8 from 9 is just 1."
"1 is a whole number and 7/8 is half of a whole."
"1 is a whole and 7/8 is not a whole."
Answer incomplete.
"When you subtract 1/9 and 7/8 you come out with a lower number."
No reason given for wrong choice.
"7/8 is almost 2 inches and 1 is only 1" (from shop class).
"I would have to guess 10 1/9 - 1 because the rest of them (the previous exercises) go along with it."
"10 1/9 - 1 = 9 1/9; you get a C.D. for 9 and 8 and the numerators would be larger; that would make the 10 1/9 smaller."
"7/8 is almost 2 wholes and 1 is only 1 whole."
"When you take 9 (of 1/9) from 8 (of 7/8) it's gonna be 1 anyway."
"If you take 7/8 from 10 1/9 you will come out less than if you take 1 from it because 7/8 is less than 1."

Which is larger: ___ 12 3/8 - 5/4 or ___ 12 3/8 - 1?

"5/4 is less than 1. If you subtract less than one you get more than if you subtract more than 1."

"5/4 is greater than 1." (13 pupils)

"If you take away 1 you get 0; if you take away 5/4 you have something left."

"12 2/4 $\left(\frac{3-5}{8-4}\right)$ is larger than 11 3/8 (12 3/8 - 1)."

No reason given for wrong choice.

"5/4 is more than a whole."

"5/4 is an improper fraction, change to a mixed number it would have a remainder, making it greater than 1."

"12 3/8 - 1 would make the 12 eleven; 12 3/8 - 5/4 you would still have 12."

"If you take 1 from 12 3/8 you'll have 11 and something; if you take 5/4 from that, you'll have 12 and something."

"12 3/8 - 1 will be taking away 1 whole; and 1 is greater than 5/4 so you'll be taking away more."

"5/4 is not larger than 1, but if you be minusing it, it would be larger than 1."

"12 3/8 = 11 13/8; 13 take away 5 is 8, so 12 3/8 - 5/4 = 11 8/8; 12 3/8 - 1 = 11 3/8 and 11 8/8 is greater than 11 3/8."

"5/4 is a mixed number and that'd make a bigger number if you broke it down than the 1."

"If you take 5/4 from this, it would not be as large as taking 1 from this."

"C.D. would be 8, then multiply numerators by what goes into that, and that would be a smaller number that go into that 12."

"12 3/8 - 1 = 11 3/8, that'll still be 12 because a fraction is not 1."

"1 is greater than 5/4 so you take away more with 1." (2 pupils)

No reason.

"You can subtract 5/4 and still have 12."

"12 - 1 = 11 and 12 3/8 - 5/4 would stay 12."

"5/4 is less than a whole."

"You will have some left because 5/4 is not a whole. In the case of the 1, the whole thing will be gone."